Methods for Computing the Power of a Matrix

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Abstract: Matrix multiplication is very important in matrix operation. In this paper, several methods of calculating matrix power are given.

Keywords: *Power; Matrix rank; Matrix decomposition; Diagonalization.*

I. INTRODUCTION

Matrix is an important and widely used concept, it is the core of linear algebra. The concept, operation and theory of Matrix run through linear algebra all the time. Matrix is a table, to master its algorithm, as table operations and number operations are not confused.

Definition 1 Let A be an n-order matrix,k a positive integer,the product of k copies of

$$A^k = \underbrace{A \cdot A \cdots A}_{k \text{ copies of } A}$$

A is called the power of k of A.

Definition 2 Let A be an n-order matrix with rank 1, then A is a product of a column vector α and a row vector β having the form as follows

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} (b_1 \quad b_2 \quad \cdots \quad b_n) = \alpha\beta^T.$$

Note The sum of the products

$a_1b_1 + a_2b_2 + \cdots + a_nb_n$ is called the trace of A, denote by tr(A)= $a_1b_1 + a_2b_2 + \cdots + a_nb_n = \alpha^T\beta = \beta^T\alpha$.

Property 1 $A^k \cdot A^l = A^{k+l}, (A^k)^l = A^{kl}$,where k,l are positive integers.

Property 2

$(AB)^k \neq A^kB^k, (A \pm B)^2 \neq A^2 \pm 2AB + B^2$.

II. COMPUTING THE POWER OF A MATRIX

Type 1 Calculating A^2 and A^3, looking patterns, then use mathematical induction to give A^k.

Example 1 Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, find A^{2021}.

Solution

$$A^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix} = 2A,$$

$$A^3 = A^2 \cdot A = 2A \cdot A = 2A^2 = 2^2A.$$

Then

$$A^k = 2^{k-1}A, A^{2021} = 2^{2020}A = 2^{2020}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Example 2 Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, find $A^k(k \geq 3)$.

Solution

$$A^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A^3 = A^2 \cdot A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

Then $A^k = 0(k \geq 3)$.

Type 2 If A=B+C, and BC=CB, then

$$A^n = (B + C)^n = \sum_{k=0}^{n} \binom{n}{k} B^kC^{n-k}.$$

Example 3 Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$, find $A^k(k \geq 2)$.

Solution $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} = E + B,$

and EB=BE, $B^k = 0(k \geq 3)$.

Then $A^k = E^k + kE^{k-1}B + \frac{k(k-1)}{2}E^{k-2}B^2$

$$=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\begin{pmatrix}0&2k&3k\\0&0&4\\0&0&0\end{pmatrix}+\begin{pmatrix}0&0&4k(k-1)\\0&0&0\\0&0&0\end{pmatrix}$$

$$=\begin{pmatrix}1&2k&4k^2-k\\0&1&4k\\0&0&1\end{pmatrix}.$$

Type 3 If A with rank 1, then $A^k=(trA)^{k-1}A$.

Example 3 $A=\begin{pmatrix}1&-1&1\\-1&1&-1\\1&-1&1\end{pmatrix}$, find A^{2021}.

Solution Since $tr(A)$=3, then

$$A^{2021}=3^{2020}A=3^{2020}\begin{pmatrix}1&-1&1\\-1&1&-1\\1&-1&1\end{pmatrix}.$$

Type 4 If A can be diagonalized, $P^{-1}AP=\Lambda$, then $A^k=P\Lambda\Lambda^kP^{-1}$.

Example 4 $A=\begin{pmatrix}0&-1&1\\2&-3&0\\0&0&0\end{pmatrix}$, find A^{99}.

Solution

Since $|\lambda E-A|=\lambda(\lambda+1)(\lambda+2)=0$, then $\lambda_1=0,\lambda_2=-1,\lambda_3=-2.$

When $\lambda_1=0$, by $-Ax$=0, obtain that $\xi_1=\begin{pmatrix}1\\1\\0\end{pmatrix}$;

When $\lambda_2=-1$, by $(-E-A)x$=0, obtain that

$\xi_2=\begin{pmatrix}1\\2\\0\end{pmatrix}$;

When $\lambda_1=-2$, by $(-2E-A)x$=0, obtain that

$\xi_3=\begin{pmatrix}3\\2\\2\end{pmatrix}$.

Assume that $P=\begin{pmatrix}1&1&3\\1&2&2\\0&0&2\end{pmatrix}$, then $A^{99}=P\Lambda^{99}P^{-1}$

$$=\begin{pmatrix}2^{99}-2&1-2^{99}&2-2^{98}\\2^{100}-2&1-2^{100}&2-2^{99}\\0&0&0\end{pmatrix}.$$

References

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