Study on Newtonian Cosmology and Its Uses

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Abstract—The basic notion of Newtonian Cosmology is noteworthy and using basic differential calculus, it can manage to derive governing equations frequently used in astrophysics and cosmology. The working knowledge of fundamental cosmology and its branches are of substantial help in learning advanced level physics. It would also be advantageous to utilize Mathematica software in detailed computations of the results.

Keywords— *Newtonian cosmology, differential calculus, governing equations, Newtonian mechanics*

I. INTRODUCTION

Most of the modern ideas in cosmology could be explained without the need to discuss General Relativity [1]. Newtonian mechanics is an approximation which works quite well for most our “earthly” needs at least when the velocity v << c, where c is the speed of light. In Newtonian mechanics, the second law states that the forces impart acceleration on the body. The rest mass generates gravitational effects in Newtonian gravity [2].

II. EQUATION OF STATE

The equation of state for non- relativistic matter and radiation is needed. In particular an expression for the rate of change of density, ρ° is needed in terms of the density ρ and pressure p [2].The first law of thermodynamics is

$$dU + dW = dQ \qquad (1)$$

where, U is the internal energy, W is the work and Q is the heat transfer. Ignoring any heat transfer and writing

dW = Fdr = pdV where F is the force, r is the distance, p is the pressure and V is the volume, then

$$dU = - pdV \qquad (2)$$

Assuming that ρ is a relativistic energy density means that the energy is expressed as

$$U = \rho V \qquad (3)$$

From which it follows that

$$\dot{U} = \dot{\rho}V + \rho\dot{V} = -p\dot{V} \qquad (4)$$

where the term on the far right hand side results from equation (2). When $V \propto r^3$ and $\frac{\dot{V}}{V} = 3\frac{\dot{r}}{r}$ thus

$$\dot{\rho} = -3(\rho + p)\frac{\dot{r}}{r} \qquad (5)$$

A. Matter

The density of matter as

$$\rho = \frac{M}{\frac{4}{3}\pi r^3} \qquad (6)$$

it follows that $\dot{\rho} \equiv \frac{d\rho}{dr}\dot{r} = -3\rho\frac{\dot{r}}{r}$ (7)

So that by comparing to equation (5) and (7), it follows the equation of state for matter is

$$-3(\rho + p)\frac{\dot{r}}{r} = -3\rho\frac{\dot{r}}{r}, \quad p = 0 \qquad (8)$$

This is the same as obtained from the ideal gas law for zero temperature.

B. Radiation

The equation of state for radiation can be derived by using radiation modes in a cavity based on analogy with a violin string [2]. For a standing wave on a string fixed at both ends

$$L = \frac{n\lambda}{2} \qquad (9)$$

Where, L is the length of the string, λ is the wavelength and n is a positive integer (n = 1,2, 3…….) Radiation travels at the velocity of light, so that

$$c = f\lambda = f\frac{2L}{n} \qquad (10)$$

where, f is the frequency. Thus substituting $f = \frac{n}{2L}c$ into Plank’s formula $U = \hbar\omega = hf$,where h is Plank’s constant, gives

$$U = \frac{nhc}{2}\frac{1}{L} \propto V^{-1/3} \qquad (11)$$

Using equation (2) the pressure becomes

$$p \equiv \frac{-dU}{dV} = \frac{1}{3}\frac{U}{V} \qquad (12)$$

Using $\rho = \frac{U}{V}$, the radiation equation of state is

$$p = \frac{1}{3}\rho \qquad (13)$$

It is customary to combine the equations of state into the form

$$p = \frac{\gamma}{3}\rho \qquad (14)$$

Where, $\gamma \equiv 1$ for radiation and $\gamma \equiv 0$ for matter. These equations of state are needed in order to discuss the radiation and matter dominated epochs which occur in the evolution of the Universe.

III. NEWTONIAN MECHANICS

A. Velocity and Acceleration Equations

The Friedmann equation, which specifies the speed of recession, is obtained by writing the total energy E as the sum of kinetic plus potential energy terms and using

$$M = \frac{4}{3}\pi r^3\rho \quad = \frac{1}{2}mr^2(H^2 - \frac{8\pi G}{3}\rho)$$

$$E = \frac{1}{2}mr^2(H^2 - \frac{8\pi G}{3}\rho) \quad = \frac{1}{2}mr^2(H^2 - \frac{8\pi G}{3}\rho) \qquad (15)$$

where the Hubble constant $H \equiv \frac{\dot{r}}{r}$, m is the mass of a test particle in the potential energy field enclosed by a gas of dust of mass M, r is the distance from the center of the dust to the test particle and G is Newton's constant. The escape velocity is just v escape = $\sqrt{\frac{2GM}{r}} = \sqrt{\frac{2G}{r}\frac{4}{3}\pi\rho r^3}$ so that the above equation can also be written

$$\dot{r}^2 = v_{escape}^2 - k' \qquad (16)$$

with $k' \equiv -\frac{2E}{m}$. The constant k' can either be negative, zero or positive corresponding to the total energy E being positive, zero or negative. For a particle in motion near the Earth this would correspond to the particle escaping (unbound), orbiting (critical case) or returning (bound) to Earth because the speed $\dot{r}$ would be greater, equal to or smaller than the escape speed v escape. Then this will be analogous to an open, flat or closed universe. Equation (15) is rearranged as,

$$\frac{2E}{mr^2} = H^2 - \frac{8\pi G}{3}\rho$$

$$H^2 = \frac{8\pi G}{3}\rho + \frac{2E}{mr^2} \qquad (17)$$

Defining $k \equiv \frac{-2E}{ms^2}$ and writing the distance in terms of the scale factor R and a constant lengths as $r(t) \equiv R(t)s$, it shows that $\frac{\dot{r}}{r} = \frac{\dot{R}}{R}$ and $\frac{\ddot{r}}{r} = \frac{\ddot{R}}{R}$, giving Friedmann equation

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2} \qquad (18)$$

which specifies the speed recession. The scale factor is introduced because in General Relativity it is space itself which expands. Even though this equation is derived for matter, it is also true for radiation [5]. In fact it is also true for vacuum, with $\Lambda \equiv 8\pi G\rho_{vac}$, where Λ is the cosmological constant and ρ_{vac} is the vacuum energy density which just replaces the ordinary density. The same equation is obtained from the general relativistic Einstein field equations. According to Guth, k can be rescaled so that instead of being negative, zero or positive it takes on the values -1, 0 or +1. From a Newtonian point of view this corresponds to unbound, critical or bound trajectories as descried above. From general relativistic point of view this corresponds to an open, flat or closed universe. In elementary mechanics the speed v of a ball dropped from a height is evaluated from the conservation of energy equation as $v = \sqrt{2gr}$, where g is the acceleration due to gravity. The derivation described above is exactly analogous to such a calculation. Similarly the acceleration of the ball is calculated as a= g from Newton's equation $F = m\ddot{r}$, where F is the force and the acceleration is $\ddot{r} \equiv \frac{d^2r}{dt^2}$. The acceleration for the universe is obtained from Newton's equation

$$-G\frac{Mm}{r^2} = m\ddot{r} \qquad (19)$$

again using $M = \frac{4}{3}\pi r^3\rho$ and $\frac{\ddot{r}}{r} = \frac{\ddot{R}}{R}$, so that it gives the acceleration equation

$$\frac{F}{mr} \equiv \frac{\ddot{r}}{r} \equiv \frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\rho \qquad (20)$$

However because $M = \frac{4}{3}\pi r^3\rho$ is used, it shows that this acceleration equation takes only for matter. In our example of the falling ball instead of the acceleration obtained from Newton's Law, it can get by taking the time derivative of the energy equation to give $a = \frac{dv}{dt} = v\frac{dv}{dr} = (\sqrt{2gr})(\frac{1}{\sqrt{2gr}}.g) = g$. Similarly for the general case one can takes the time derivative of equation (18) (valid for matter and radiation)

$$\frac{d}{dt}\dot{R}^2 = 2\dot{R}\ddot{R} = \frac{8\pi G}{3}\frac{d}{dt}(\rho R^2) \qquad (21)$$

Upon using equation (15) the acceleration equation is obtained as

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p) = -\frac{4\pi G}{3}(1+\gamma)\rho \qquad (22)$$

which reduces to equation (20) for the matter equation of state ($\gamma = 0$).

B. Cosmological Constant

In both Newtonian and relativistic cosmology the universe is unstable to gravitational collapse. Both Newton and Einstein believed that the universe is static. In order to obtain this Einstein introduced a repulsive gravitational force, called the cosmological constant, and Newton is done exactly the same thing, believed the universe to be finite [3]. In order to obtain a possibly zero acceleration a positive term (conventionally taken as $\frac{\Lambda}{3}$ is added to the acceleration equation (22) as

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \qquad (23)$$

which ,with the proper choice of Λ gives the required zero acceleration for a static universe. Again exactly the same equation is obtained from the Einstein field equation. Identifying the force from

$$\frac{\ddot{r}}{r} = \frac{\ddot{R}}{R} \equiv \frac{F_{repulsive}}{mr} \equiv \frac{\Lambda}{3} \qquad (24)$$

and using $F_{repulsive} = \frac{\Lambda}{3}mr \equiv \frac{-dV}{dr}$ gives the potential as

$$V_{repulsive} = -\frac{1}{2}\frac{\Lambda}{3}mr^2 \qquad (25)$$

which is just a repulsive simple harmonic oscillator. Substituting this into the conservation of energy equation

$$E = T + V = \frac{1}{2}m\dot{r}^2 - G\frac{Mm}{r} - \frac{1}{2}\frac{\Lambda}{3}mr^2$$

$$= \frac{1}{2}mr^2(H^2 - \frac{8\pi G}{3}\rho - \frac{\Lambda}{3}) \qquad (26)$$

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2} + \frac{\Lambda}{3} \qquad (27)$$

Equation (23) and (27) constitute the fundamental equations of motion that are used in all discussions of Friedmann models of the universe. Exactly the same equations are obtained from the Einstein field equations. The solution is obtained by noting that whereas gravity is an inverse square law for point masses M and m separated by a distance r as given by $F = G\frac{Mm}{r^2}$, if one of the masses is a continuous mass distribution represented by a density then $F = G\frac{4}{3}\pi\rho m r$. Finally, the cosmological constant in terms of a vacuum energy density as $\Lambda \equiv 8\pi G\rho_{vac}$ so that the velocity and acceleration equation become

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2} + \frac{\Lambda}{3} = \frac{8\pi G}{3}(\rho + \rho_{vac}) - \frac{k}{R^2} \quad (28)$$

$$\frac{\dot{R}}{R} = -\frac{4\pi G}{3}(1+\gamma)\rho + \frac{\Lambda}{3} = -\frac{4\pi G}{3}(1+\gamma)\rho = \frac{8\pi G}{3}\rho_{vac} \quad (29)$$

C. Einstein Static Universe

Although we have noted that the cosmological constant provides repulsion, it is interesting to calculate its exact value for a static universe [4].The Einstein static universe requires R = R_0 = constant and thus $\dot{R} = \ddot{R} = 0$. The case $\ddot{R}$ = 0 will be examined first. From equation (23) this requires that

$$\Lambda = 4\pi\ G(\rho + 3p) = 4\pi\ G(1+\gamma)\rho \quad (30)$$

If there is no cosmological constant (Λ = 0) then either ρ = 0 which is an empty universe, or p = $-\frac{1}{3}\rho$, which requires negative pressure. Both of these alternatives were unacceptable to Einstein and therefore concluded that a cosmological constant was present, ie $\Lambda \neq 0$. From equation (30) this implies

$$\rho = \frac{\Lambda}{4\pi G(1+\gamma)} \quad (31)$$

and because ρ is the positive this requires a positive Λ. Substituting equation (31) into equation (26) , we get

$$\Lambda = \frac{3(1+\gamma)}{3+\gamma}\left[\left(\frac{\dot{R}}{R_0}\right)^2 + \frac{k}{R_0^{\ 2}}\right] \quad (32)$$

Now imposing $\dot{R} = 0$ and assuming a matter equation of state $(\gamma = 0)$ implies $\Lambda = \frac{k}{R_0^2}$. However the requirement that Λ be positive force k = +1 giving $\Lambda = \frac{1}{R_0^2} = \text{constant}$ (33)

Thus the cosmological constant is not only old value but rather simply the inverse of the scale factor squared' where the scale factor has a fixed value in this static model [4].

D. Uses in Conservation Laws

Maxwell equations imply the conservation of charge, so velocity and acceleration equations imply conservation of energy. The energy- momentum conservation equation is derived by setting the covariant derivative of the energy momentum tensor equal to zero. The same result is achieved by taking the time derivative of equation [4]. The result is

$$\dot{\rho} + 3(\rho + p)\frac{\dot{R}}{R} = 0 \quad (34)$$

This is identical to equation (15) illustrating the interesting connection between thermodynamics and General Relativity. However, the velocity and acceleration equations can be obtained directly from the Einstein field equations. Thus, the Einstein equations imply the thermodynamic relationship in the above equation. The above equation may also be written as

$$\frac{d}{dt}(\rho R^3) + p\frac{dR^3}{dt} = 0 \quad (35)$$

And equation (14), $3(\rho + p) = (3+\gamma)\rho$, it follows that

$$\frac{d}{dt}(\rho R^{3+\gamma}) = 0 \quad (36)$$

Integrating the above equation, $\rho = \frac{c}{R^{3+\gamma}}$ (37)

Where, c is a constant. This show that the density falls as $\frac{1}{R^3}$ for matter and $\frac{1}{R^4}$ for radiation as expected. The different form of the above equations are as follows. From equation (31)

$$\rho' + 3(\rho + p)\frac{1}{R} = 0 \quad (38)$$

Where, prime shows derivatives with respect to R, i.e. $x' \equiv \frac{dx}{dR}$. Alternatively

$$\frac{d}{dR}(\rho R^3) + 3pR^3 = 0 \quad (39)$$

$$\frac{1}{R^{3+\gamma}}\frac{d}{dR}(\rho R^{3+\gamma}) = 0 \quad (40)$$

which is consistent with equation (34).

E. Age of the Universe

Recent measurements made with the Hubble space telescope have determined that the age of the universe is younger than globular clusters. A possible resolution to this paradox involves the cosmological constant. Writing equation (28) as

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi\ G}{3}\rho - \frac{k}{R^2} + \frac{\Lambda}{3}, \Lambda = 8\pi\ \ G\rho_{vac}$$

$$\dot{R}^2 = \frac{8\pi G}{3}(\rho + \rho_{vac})R^2 - k \quad (41)$$

the value of k is

$$k = \frac{8\pi G}{3}(\rho_0 + \rho_{0\ vac})R_0^2 - H_0^2 R_0^2 \quad (42)$$

with $H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2$. The values of quantities have been denoted with a subscript 0. Substituting equation (39) into equation (38) yields

$$\dot{R}^2 = \frac{8\ \pi G}{3}(\rho + \rho_{vac})R^2 - \frac{8\ \pi G}{3}(\rho_0 + \rho_{0vac})R_0^2 - H_0^2R_0^2$$

$$\dot{R}^2 = \frac{8\pi G}{3}(\ \rho R^2 - \rho_0 R_0^2 + \rho_{vac}\ R^2\rho_{0vac}\ R_0^2) - H_0^2R_0^2$$

Integrating gives the expansion age

$$T_0 = \int_0^{R_0} \frac{dR}{\dot{R}} = \int_0^{R_0} \frac{dR}{\sqrt{\frac{8\pi G}{3}(\rho R^2 - \rho_0 R_0^2 + \rho_{vac} R^2 - \rho_{0vac} R_0^2) - H_0^2 R_0^2}}$$

For the cosmological constant $\rho_{vac} = \rho_{0\ vac}$ and because $R_2 < R_{02}$ then a nonzero cosmological constant will give an age larger than would have been obtained were it not present. The inclusion of a cosmological constant gives an age which is larger than if no constant were present.

CONCLUSION

In this work attempts are made to give the notion of Newtonian cosmology based entirely on Newtonian mechanics. The equations describing the velocity (called the Friedmann equation) and acceleration of the universe are derived from Newtonian mechanics and also the cosmological constant is introduced within a Newtonian framework. The equations of state are also derived in a very simple way. The fundamental equations of motion that are used in all discussions of Friedmann models of the universe are also presented. Some applications of conservation laws such as conservation of charge are utilized in Maxwell equation. Energy-momentum conservation equation can be obtained by taking the time derivative of velocity and acceleration equation and interesting connection between thermodynamics and General Relativity has been given.

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