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On Measurability and Measures of Sets and Functions in R

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Abstract: In this paper, we analyse the concept of measurability of sets of real numbers, with the meaning of measures of specific sets. The measurability of a function follows from the measurability of the domain.

I. INTRODUCTION

The theory of set measure was developed from some basic axioms and definitions. The focus of sets here will be the field of real numbers, $R = (-\infty, \infty)$. A point represents a number on real number line. $-\infty \leftrightarrow \infty$.

Measure of Subset $A \subseteq R$

Let A be an open interval, $A = (a, b)$. The measure of (a, b) is its length $b - a$. Also, the measure of the closed interval, $A = [a, b]$ is equal to $b - a$. The set can be written as a union of open sub-intervals, I_i, $i = 1,2,3,\cdots$

Then

$$A = \bigcup_{i=1}^{\infty} I_i$$

We denote the measureof A by $|A|$

Then,

$$|A| = \sum_{i=1}^{\infty} |I_i|$$

If I_1 and I_2 are arbitrary intervals, then it is obvious that

$$|I_1 \cup I_2| \leq |I_1| + |I_2|$$

Thus,

$$\left|\bigcup_{i=1}^{\infty} I_i\right| \leq \sum_{i=1}^{\infty} |I_i|$$

We now consider a more general definition of measure of a set A. First, we consider the following definition. Given two non-empty sets $A$and B, the set $B - A$ is the of element of B which are not in A.

II. MEASURABILITY

A set A is said to be measurable if for every $\epsilon > 0$, there exist open sets O and U such that

$$O \supset A,\ U \supset O - A, \text{ and } |U| < \epsilon.$$

Clearly, every open or closed set is measurable.

The Labesque Measure of a Set A

The Labesque measure of a measurable set A is the greatest lower bound of the measure of its supper sets.

Set of Measure Zero

From the definition, a point has the measurezero**.** Consider the closed interval [0.1]. We constract a set of open sets from $[0,1]$, as follows. Let U_1 be the open interval $(\frac{1}{3}, \frac{2}{3})$, which is of measure $\frac{1}{3}$. Removing $U_1$from $[0,1]$, there remain the two closed intervals $[0, \frac{1}{3}]$ and $\left[\frac{2}{3}, 1\right]$. Now out of each we remove the middle open interval of length $\frac{1}{3^2}$, there sum being equal to $\frac{2}{3^2}$. From the remains four intervals we remove from each, the middle open interval, each of length $\frac{1}{3^3}$, and adding to give the measure $\frac{2}{3^3}$. And so on.

Thus, at each k stage, we remove an open set ofmeasure equal to $\frac{2^{k-1}}{3^k}$; $k = 1,2,\cdots$

Thus, from the interval $[0,1]$, for each k we removed an element of a sequence of open interval of measure equal $\frac{2^{k-1}}{3^k}$; $k = 1,2,\cdots$

Then, summing all the removed open intervals, we obtain the series:

$$\sum_{k=1}^{n} \frac{2^{k-1}}{3^k} = \frac{1}{3}\left[\frac{1 - \left(\frac{2}{3}\right)^n}{1 - \frac{1}{3}}\right]$$

Then, in the long run, we obtain an infinite union of open sets

$$A = \bigcup_{i=k}^{\infty} \bigcup_{k=1}^{2^{k-1}} I_{i,k} \qquad \text{..............................} (1)$$

The measure of the set (1) is equal to

$$\lim_{k \to \infty} \left\{\frac{1}{3}\left[\frac{1 - \frac{2^k}{3^{k+1}}}{1 - \frac{2}{3}}\right]\right\} = 1$$

Thus, $|A| = 1$

The complement of the set A is a closed set, and this consists of the boundary points of the removed open intervals.

The measure of this closed set is zero, since, if $H = [0,1] - A$,

Then,

$$\begin{aligned} |H| &= |[0,1] - A| \\ &= |[0,1]| - |A| \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

Thus, $|H| = 0$.

Thus, the measure of union of an infinite set of points is zero. Hence, the measure of the set of rational numbers in the closed interval [0,1] has measure zero.

Now, by definition the measure of [0,1] is equal to 1.

Now the interval [0,1] consists of two sets of n umbers rational and irrational.

Then, if R and I denote the set of rational and irrational numbers in [0,1], we have

$$[0,1] - R = I$$

Then,

$$|[0,1] - R| = |I|$$

which implies

$$|[0,1]| - |R| = |I|$$

$$1 - 0 = |I|$$

That is

$$|I| = 1$$

Thus, the measure of the set of irrational numbers in [0,1] is equal to 1, i.e., the measure of the interval. Hence, in general the measure the measure of the set of irrational numbers in any interval I, is $|I|$. Further, the set of the irrational numbers is open, since its complement$R$ is closed.

Measurability of a Function, *f*

A function, f, is said to be measurable if there exist $a \in R$, such that the set of numbers

$$\{\ x : f(x) < a\ \}$$

is measurable.

We state, without proof the following theorems.

Theorem 1. Each of the following conditions is a necessary and sufficient condition for a function, $f(x)$ to be measurable:

(i) The set $\{ x : f(x) < r \}$ is measurable for every rational number r.

(ii) The set $\{ x : f(x) \leq r \}$ is measurable for every rational number r.

Theorem 2. If $f(x)$ is measurable, then the sets

$$\{ x : f(x) \leq a \},\ \ \{ x : f(x) > a \}\text{and}\{ x : f(x) = a \}$$

are measurable for every $a \in R$.

Theorem 3. The constant function $f(x) = a$ is measurable.

Theorem 4. The finite unions and intersections of measurable functions are measurable.

Theorem 5. If the function $f(x)$ is measurable and the equation $f(x) = g(x)$ holds except at points of measure zero, then $g(x)$ is measurable.

The definition of measurability of function shows that every well-defined and bounded function is measurable.

SUMMARY AND CONCLUSION

We have, in this paper, analysed the concept of measurability and measure of sets and functions in R. The measurability of a function is related the measurability of the domain of the function.

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