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On Suns, Moons and Best Approximation in M-Space

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Abstract: In this paper we discuss suns and moons in M-spaces and characterize these via best approximation thereby extending corresponding known results in normed linear spaces to M-spaces.

I. INTRODUCTION

The concept of a sun in Approximation Theory was first introduced in normed linear spaces by Klee(1953)but the terminology 'sun' was proposed by Effimov and Steckin(1958). We recall that a set V in a normed linear space X is a sun iff whenever $v_0 \in$ V is a best approximation to some element $x \notin$ V then v_0 is a best approximation to every element on the ray from v_0 through x. Since every convex set in a normed linear space has this property, a sun may be regarded as a generalization of a convex set. L.P. Vlasov,who developed the concept further in Vlasov(1961),showed that in a smooth Banach space every proximinal sun is convex. The concept of a moon, which is a generalization of sun, was introduced by Amir and Deutsch(1972) and their special interest was in determining those normed linear spaces in which every moon is a sun. Knowing such spaces is quite useful as it is much easier to verify that a given set is a moon than verify it is a sun.

A. *Normed Linear Space*

Let V be a linear space. We recall that a norm is a function from V into non-negative real numbers. This function is written ‖. ‖ and satisfies the following three properties:

1) $\| v \| \geq 0$ with equality if and only if v=0
2) $\| v \| = | \lambda | \| v \|$ for any scalar λ
3) $\| v + w \| \leq \| v \| + \| w \|$ (the triangle inequality)

Then, (v, ‖. ‖) is called *normed linear space.*

The *norm* gives us a notion of distance in v. if w, v ϵ V, then the distance from w to v (or v to w) is $\| v\text{-}w \|$

B. *Convex*

A set S, in a linear space is *convex* .if $s_1, s_2 \epsilon S$ implies that $\lambda_1 s_1 + \lambda_2 s_2 \epsilon S$

if λ_1 andλ_2 are non negative and

$\lambda_1 + \lambda_2 = 1$

If S is empty or consists of one point, then it is clearly *convex.*

C. *Best Approximation*

Let G be a nonempty subset of a real normed linear space X and let an element fϵ X be given. The problem of *best approximation* is to determine an element $g_f \epsilon G$ such that

$$\| f\text{-} g_f \| = \inf_{g \epsilon G} \|f - g\|$$

such an element is called a *best approximation* to f from G ,and

$$d(f, G) = \inf_{g \epsilon G} \|f - g\|$$

is called the *minimal deviation* of f from G.

The set of all elements $g_0 \epsilon$ G that are called best approximation to x ϵ X is

$P_G(x) = \{ g_0 \epsilon G: \| x - g_0 \| \leq \| x - g \|$ for all $g \epsilon G \}$

D. *M-Space*

A metric space (X, d) in which for every x, y ∈ X and for every t, $0 \leq t \leq 1$ there exists exactly one point z ∈ X such that d(x, z) = (1 − t)d(x, y) and d(z, y) =td(x, y) is called an M-space

E. *Cone*

A subset V of an M-space (X, d) is said to be a cone with vertex v_0 if $G(v_0, y, -) \subseteq V$ whenever $y \in V$.

For $v_0 \in V$, $P_v^{-1}(v_0) = \{x \in X : v_0 \in P_v(x)\}$. It is easy to prove that if $x \in P_v^{-1}(v_0)$ then

$x_\lambda \in P_v^{-1}(v_0)$ for every $x_\lambda \in G[v_0, x]$ i.e.$v_0 \in P_v (x_\lambda)$. On the other hand, v_0 may not be in $P_v (x_\lambda)$ for $x_\lambda \in G_1(v_0, x, -)$.

F) *Proximinal Set*

If $P_G(x)$ contains at least one element, then the subset G is called a *proximinal set.*

In other words, if $P_G (x) \neq \emptyset$ then G is called a *proximinal set*

The term *proximinal set* (is a combination of proximity and maximal)

G) *Solar Point*

If V is a proximinal subset of an M-space (X, d) , a point $v_0 \in V$ is called a solar point

(Figure) of V if $x \in P_v^{-1}(v_0)$ implies $x_\lambda \in P_v^{-1}(v_0)$ for every $x_\lambda \in G_1(v_0, x, -)$. The set V is called a sun (see Fig.4.2) if for each $x \in X \setminus V$, every $v_0 \in P_v(x)$ is a solar point of V i.e. for all $v_0 \in P_v(x)$, $v_0 \in P_v(z)$ for all $z \in G_1(v_0, x, -)$.

H. *Example*

Consider the normed linear space R^2 with supremum-norm and

$V = \{(x,y) \epsilon R^2: x \geq 0 \text{ or } y \geq 0\}$. Then V is a sun

I. *Example*

The set $V = \{(x, y) \in R^2 : x^2 + 4y^2 \leq 1\}$ in Euclidean 2-space $R^2$is a sun.

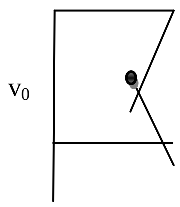

Figure 1: Solar point

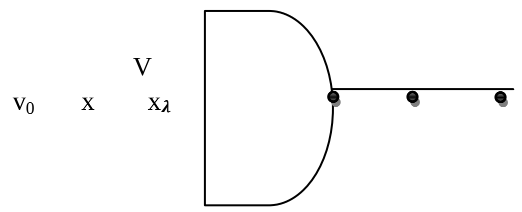

Figure 2: A set which is a sun

J. Lunar Point

Let V be a subset of an M-space (X, d) . A point $v_0 \in V$ is called a lunar point if $x \in X$ and $K(v_0, x) \cap V \neq \emptyset$ imply $v_0 \in \overline{K(v_0, x) \cap V}$ where $K(v_0, x) = \cup \{B(z, d(z, v_0)), z \in G_1(v_0, x,-)\}$. The set V is called a moon if each of its point is lunar.

K. Example

The set $V = \{(x, y) \in R^2 : x^2 + 4y^2 \geq 1\}$ in Euclidean 2-space R^2 is a moon.

We shall see that each sun in an M-space is a moon. However, the converse is not truEehe following two theorem give necessary and sufficient condition for a proximinal subsets of an M-space to a sun. we have:

II. THEOREMS

A. Theorem

A proximinal subset V of an M-space (X, d) is a sun if and only if for any $v_0 \in V$, the set $P_v^{-1}(v_0)$ is a cone with vertex v_0.

Proof

Suppose V is a sun and $x \in P_v^{-1}(v_0)$ i.e. $v_0 \in P_v(x)$.

We want to show that $G(v_0, x,-) \subseteq P_v^{-1}(v_0)$.

Since $v_0 \in P_v(x)$ and V is a sun, $v_0 \in P_v(z)$ for all $z \in G_1(v_0, x,-)$and consequently for all $z \in G(v_0, x,-)$ i.e. $z \in P_v^{-1}(v_0)$ for all $z \in G(v_0, x,-)$ i.e.$P_v^{-1}(v_0)$ is a cone with vertex v_0. Conversely,

let $x \in X \setminus V$ and $y \in P_v(x)$ i.e.

$x \in P_v^{-1}(y)$ where $y \in V$.

Since $P_v^{-1}(y)$is a cone with vertex y,

$G(y, x,-) \subseteq P_v^{-1}(y)$

i.e. $y \in P_v(z)$ for all $z \in G(y, x,-)$.

Hence V is a sun.

B. Theorem

A proximinal subset V of an M-space (X, d) is a sun if and only if for any$v_0 \in V$ and $x \in P_v^{-1}(v_0)$, $K(v_0, x) \cap V = \emptyset$.

Proof

Suppose V is a sun.

Let $v_0 \in V$ and $x \in P_v^{-1}(v_0)$.

Since $v_0 \in P_v(x)$ and V is a sun, $v_0 \in P_v(z)$ for all $z \in G(v_0, x,-)$.

To show $K(v_0, x) \cap V = \emptyset$.

Suppose $u \in K(v_0, x) \cap V$

i.e. $u \in B(z, d(z, v_0))$ for some

$z \in G_1(v_0, x,-)$

i.e. $d(z, u) \leq d(z, v_0)$ and so $v_0 \notin P_v(z)$ as $u \in V$, a contradiction.

Therefore $K(v_0, x) \cap V = \emptyset$.

For the converse part,

suppose V is not a sun.

Then there exists $x \in X \setminus V$ and$v_0 \in P_v(x)$ such that $v_0 \notin P_v(z)$ for some

$z \in G(v_0, x,-)$.

Then $d(z, v_1) \leq d(z, v_0)$ where $v_1 \in P_v(z)$ i.e. $v_1 \in B(z, d(z, v_0))$ for some

$z \in G(v_0, x,-)$. i.e $v_1 \in K(v_0, x)$. Also $v_1 \in V$ and therefore $K(v_0,x) \cap V \neq \emptyset$, a contradiction.

Hence V is a sun.

C. Lemma

Let V be a subset of an M- space (X, d) then , $K(v_0, x) = K(v_0, y)$ for all $y \in G[v_0, x]$, where $x \in X$, $V \subset X$ and$v_0 \in P_v(x)$.

Proof

$K(v_0, x) = \cup \{B(z_1,d(z_1,v_0)),z_1 \in G_1(v_0, x,-)\}$,

$K(v_0, y) = \cup \{B(z_2, d(z_1, v_0)), z_2 \in G_1(v_1, y,-)\}$.

Let $z \in K(v_0, x)$ then $z \in B(z_1, d(z_1, v_0))$ for some $z_1 \in G_1(v_0, x,-)$.

Now any $z_1 \in G_1(v_0, x,-)$ is also a point on $G_1(v_0, y,-)$ i.e. $z_1 = z_2$ for some

$z_2 \in G_1(v_0, y,-)$

i.e. $z \in \cup \{B(z_2, d(z_2, v_0)), z_2 \in G_1(v_0, y,-)\}$. Therefore,

(1) $K(v_0, x) \subseteq K(v_0, y)$

Let $z \in K(v_0, y)$then $z \in B(z_2, d(z_2, v_0))$ for some $z_2 \in G_1(v_0, y,-)$.

If $z_2 \in G_1(v_0, x,-)$ then $z \in K(v_0, x)$ and so $K(v_0, y) \subseteq K(v_0, x)$.

If $z_2 \in G[y, x]$,

consider $z' \in G_1(v_0, x,-)$.

Then

$d(z, z') \leq d(z, z_2) + d(z_2, z')$

$\leq d(z_2, v_0) + d(z_2, z')$

$= d(z', v_0)$.

Therefore $z \in B(z', d(z', v_0))$ and so

$z \in K(v_0, x)$.

Consequently

(2) $K(v_0, y) \subseteq K(v_0, x)$.

(1) and (2) imply $K(v_0, x) = K(v_0, y)$.

The following theorem shows that we may assume in the definition of lunar point that x has v_0 as a best approximation from V.

D. Theorem

Let V be a subset of an M-space (X, d) and $v_0 \in V$. Then the following are equivalent: v_0 is a lunar point whenever v_0 is a best approximation to x with $K(v_0, x) \cap V \neq \emptyset$ then $v_0 \in \overline{K(v_0, x) \cap V}$.

Proof

$\Rightarrow$(ii) is trivial.

$\Rightarrow$(i).

Let $x \in X$ and $K(v_0, x) \cap V \neq \emptyset$.

To show $v_0 \in \overline{K(v_0, x) \cap V}$.

If v_0 is a best approximation to x then by (ii), $v_0 \in \overline{K(v_0, x) \cap V}$.

If v_0 is not a best approximation to x then two cases arise:

(a) v_0 is not a local best approximation to x,

(b) v_0 is a local best approximation to x.

Case (a):

If v_0 is not a local best approximation to x i.e. for all $\epsilon \geq 0$ there exists $v_\epsilon \in V$

Such that $d(v_\epsilon, v_0) \leq \epsilon$ and $d(v_\epsilon, x) \leq d(v_0, x)$.

Then $v_\epsilon \in B(x, d(v_0, x)) \subset K(v_0, x)$.

Therefore every neighbourhood of v_0 contains an element v_ϵ of $K(v_0, x) \cap V$

other than v_0 i.e. v_0 is a limit point of $K(v_0, x) \cap V$ and so $v_0 \in \overline{K(v_0, x) \cap V}$.

Hence v_0 is a lunar point.

Case (b):

If v_0 is a local best approximation to x

i.e. v_0 is a best approximation to x from

$V \cap B(v_0, \epsilon)$ for some $\epsilon \geq 0$.

Let $z \in G[v_0, x]$ such that $d(z, v_0) \leq \frac{\epsilon}{2}$

then by Lemma c $K(v_0, z) = K(v_0, x)$ and v_0 is a best approximation to z from V.

So (ii) implies

$v_0 \in \overline{K(v_0, z) \cap V} = \overline{K(v_0, x) \cap V}$ and

therefore v_0 is a lunar point.

E. Corollary

Every sun in an M-space is a moon.

Proof

Let V be a sun.

Suppose V is not a moon

i.e. there exists $v_0 \in V$ which is not a lunar point

i.e. v_0 is a best approximation to $x \in X$ with $K(v_0, x) \cap V \neq \emptyset$

but $v_0 \notin \overline{K(v_0, x) \cap V}$.

As V is a sun, Theorem b implies $K(v_0, x) \cap V = \emptyset$

whenever v_0 is a best approximation to $x \in X$.

Since these two statements are contradictory, the result follows.

CONCLUSION

In this paper discussed about the sun and moon in M-Spaces and characterized these via best approximation thereby extending corresponding known results in normed linear spaces to M-spaces. That is the sun and moon concepts in M-spaces also called strongly convex spaces and extended some of the results proved in normed linesr spaces by Amir and Deutch (1972) and Mhaskar and pai (2000) to M-spaces

References

[1] H.Mazaheri ,"The Nearest Points in Normed Linear Spaces" applied Mathematical Science, Vol.2, 2008, no.16.763-766.

[2] A.L. Garkavi, The theory of best approximation in normed linear space. Mathematical analysis 1967, Moscow (1969), 75-132n[Russian].

[3] P.K.Jain,O.P.Ahuja and Khalil Ahmad."Functional analysis". New Age International(P)Limited,1995.

[4] Holmes,R.A A course on Optimization and best approximation,(CLNM 257,Springer)1972 [5]Ivan Singer.The theory of best approximation and functional analysis(1974).

[5] Part III Lent term, Approximation theorem-lecture 4,Existence and uniqueness of bestapproximation.2005