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A Study on Sarkovskii's Theorem in Chaos With its Applications

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Abstract- This paper deals with chaos periodic properties in multi fluctuations. Also discuss about Sarkovskii's theorem with its converse proof. In the application part we have given the analysis of stress detection in the focus of control systems in short sighted eye and phase space reconstruction.

Keywords- *Sensitive Initial Condition, Sarkovskii's, theorem, Lyapunov Exponent, Micro fluctuations, Embedding.*

I. INTRODUCTION

Chaos theory is the study of complex, nonlinear and dynamic systems. The field was pioneered by Lorenz who was studying the dynamics of turbulent flow in fluids. At the limit, chaotic systems can become truly random. Mathematically, chaotic systems are represented by differential equations that cannot be solved, so that we are unable to calculate the state of the system at a specific future time't'. Chaotic systems never return to the same exact state, but the outcomes are bounded and create patterns. A non linear difference equation in one variable can generate chaos and an Ordinary.

Differential Equation in three variables can generate chaos. The major achievements of Chaos theory is its ability to demonstrate how a simple set of deterministic relationships can produce patterned but unpredictable outcomes. The applications of chaos theory has generated in so many fields.

An early proponent of chaos theory was *Henri Poincare*. In the 1880s, while studying the three-body problem, he found that there can be orbits that are non-periodic, and yet not forever increasing nor approaching a fixed point. In 1898 *Jacques Hadamard*published an influential study of the chaotic motion of a free particle gliding frictionless on a surface of constant negative curvature, called "*Hadamard's billiards*".

Hadamard was able to show that all trajectories are unstable, in that all particle trajectories diverge exponentially from one another, with a positive Lyapunov exponent.

Chaos theory began in the field of *ergodic theory*. The main catalyst for the development of chaos theory was the electronic computer. Much of the mathematics of chaos theory involves the repeated iteration of simple mathematical formulas, which could be impractical to do by hand. Electronic computers made these repeated calculations practical, while figures and images made it possible to visualize these systems.

Edward Lorenz was an early pioneer of the theory. His interest in chaos came about accidentally through his work on *weather prediction* in 1961.Lorenz discovered that small changes in initial conditions produced large changes in long-term outcome.

II. CHAOS

The name 'chaos theory' leads to believe that mathematicians have discovered some new and definitive knowledge about utterly random and incomprehensible phenomena. Chaos theory is the qualitative study of unstable a-periodic behavior in deterministic nonlinear dynamical systems. Chaos theory concerns deterministic systems whose behaviors in principle can be predicted. Chaotic systems are predictable for a while and then appear to become random. Chaos theory is a branch of mathematics focused on the behavior of dynamical systems that are highly sensitive to initial conditions.

The amount of time that the behavior of a chaotic system can be effectively predicted depends on three things: how much uncertainty can be tolerated in the forecast, how accurately its current state can be measured and a time scale depending on the dynamics of the system called the Lyapunov time.

"When the present determines the future but the approximate present does not approximately determine the future".

A. Chaos

Chaos is the aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on the initial condition.

B. Aperiodic Long Term

Aperiodic long-term behavior means that trajectories do not settle down to fixed points, periodic orbits or quasi-periodic orbits as t→ ∞. Deterministic means that the irregular behavior of the system arises from its being nonlinear, not from added external noise.

Chaos theory states that the apparent randomness of complex systems within it, and on programming at the initial point is known as initial conditions. The butterfly effect of chaos theory explains a small change of a deterministic nonlinear system. Many natural systems with chaotic behaviors like weather and climate. The application of chaos theory includes algorithmic trading, cryptography, psychology and robotics.

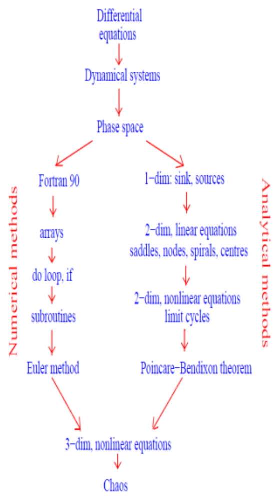

Figure 1: Chaos from Differential Equations

C. Chaotic Dynamics

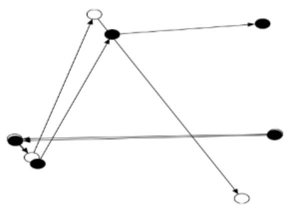

Figure 2: Chaotic Dynamics

In common, "chaos" means "a state of disorder".

To classify a dynamical system as chaotic, it must have those properties:

- It must be sensitive to initial conditions.
- It must be topologically mixing.
- It must have dense periodic orbits.

D. Sensitivity To Initial Conditions

It means that each point in a chaotic system is arbitrarily closely approximated by other points with significantly different future paths, or trajectories. Sensitivity to initial condition is known as *butterfly effect.*

A consequence of sensitivity to initial conditions is that if we start with a limited amount of information about the system, then beyond a certain time the system is no longer predictable. This is most familiar in the case of weather, which is generally predictable.

A positive *Maximal Lyapunov Exponent (MLE)*is usually taken as an indication that the system is chaotic

E. Toplogical Mixing

Topological mixing (or topological transitivity) means that the system evolves over time so that any given region or open set of its phase space eventually overlaps with any other given region. Topological mixing is often omitted from popular accounts of chaos, which equate chaos with only sensitivity to initial conditions. However, sensitive dependence on initial conditions alone does not give chaos. Indeed, it has extremely simple behavior: all points except 0 tend to positive or negative infinity.

F. Density Of Periodic Orbits

For a chaotic system to have dense periodic orbits means that every point in the space is approached arbitrarily close by periodic orbits. The one-dimensional logistic map defined by $x \to 4x(1-x)$ is one of the simplest systems with density of periodic point.

III. SARKOVSKII ORDERING

Defining the sarkovskii ordering on the natural numbers:

$3 \rhd 5 \rhd 7 \rhd \ldots \rhd 2.3 \rhd 2.5 \rhd 2.7 \rhd \ldots \rhd 2^2.3 \rhd 2^2.5 \rhd 2^2.7 \rhd \ldots \rhd 2^3.3 \rhd 2^3.5 \rhd 2^3.7 \rhd \ldots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1.$

That is, all odd numbers greater than one in increasing order, followed by 2 times those numbers, then by 2^2 times then 2^3, and so on.

This exhausts all the natural numbers with the exception of the powers of two in decreasing order.

A. Sarkovskii's Theorem

Suppose f: $R \to R$ is continuous. Suppose that f has a periodic point of period k and that k precedes l in the sarkovskii ordering. Then f also has a periodic point of prime period k.

PROOF:

Remark

- If f has a periodic point whose period is not a power of two, then f necessarily has infinitely many period points. Conversely, if f has only finitely many periodic points, then they all necessarily have periods which are powers of two.
- Period 3 is the greatest period in the sarkovskii ordering and therefore implies the existence of all other periods.

Case (i): n is odd.

Assume that f has a periodic point x of period n where n is odd and n-1.

Suppose that f has no periodic point of odd period less than n.

Let $x_1, x_2, \ldots, x_n$ be the points on the orbit of x.

Note that, f permutes the x_i.

Clearly,$f(x_n)<x_n$. (1)

Choose i be the largest for which $f(x_i) > x_i$.

Consider an interval $I_1 = [x_i, x_{i+1}]$.

$f(x_{i+1})<x_{i+1}$[by (1)]

It follows that $f(x_{i+1}) \leq x_i$ and so we have $I_1 \subset f(I_1)$.

$\therefore I_1 \to I_1$.

Since x does not have period 2, so it can't be $f(x_{i+1}) = x_i$ and $f(x_i) = x_{i+1}$.

Thus, $f(I_1)$ contains at least one interval of the form $[x_j, x_{j+1}]$

Let O_2 be the union of intervals of the form $[x_j, x_{j+1}]$ which is covered by $f(I_1)$.

Hence $I_1 \subset O_2$ but $O_2 \neq I_1$.

If I_2 be any interval in O_2 of the form $[x_j, x_{j+1}]$, then $I_1 \to I_2$.

Now, let O_3 be the union of intervals of the form $[x_j, x_{j+1}]$ which is covered by the image of some interval in O_2.

Continuing in this way, we get O_{l+1} be the union of intervals, which is covered by the image of some interval in O_l. Thus, if I_{l+1} is any interval in O_{l+1} then there is a collection of intervals $I_2, \ldots, I_l$ with $I_j \subset O_j$ satisfying $I_1 \to I_2 \to \ldots \to I_l \to I_{l+1}$.

$\Rightarrow O_l$ form an increasing union of intervals.

Since there are only finitely many x_i, it follows that there is k for which $O_{l+1} = O_l$.

For this l, O_l must contain all intervals of the form $[x_j, x_{j+1}]$.Otherwise x will have period less than n.

<u>*Claim*</u>*:*There is atleast one interval $[x_j, x_{j+1}]$ different from I_1 in some O_k whose image covers I_1.

Since there are more x_i's on one side of I_1 than on the other (n is odd).

Hence some x_i's must change sides under the action of f and some must not. Therefore, there is atleast one interval whose image covers I_1.

Now, consider the chain of intervals $I_1 \to I_2 \to_{\ldots} \to I_k \to I_l$ where each I_l is of the form $[x_j, x_{j+1}]$ for some j and $I_2 \neq I_1$.

By claim, there is atleast one such chain.

Choose k be the smallest for which $I_1 \to I_2 \to \ldots I_k \to I_l$ is the shortest path from I_1 to I_1 except $I_1 \to I_1$.
Now, if $l >$ m-1 then one of loop either $I_1 \to I_2 \to \ldots \to I_k \to I_1$ or $I_1 \to I_2 \to \ldots \to I_k \to I_1 \to I_1$ gives a fixed point of f^m with m odd and m < k.

Since $I_1 \cap I_2$ consists of only one point whose period is greater than m.

$\Rightarrow$The point must have prime period less than k.

$\therefore$ k = n-1.

For any j $> l+1$, we have I_l does not tends to I_j.

$\Rightarrow$The orbit of x must be ordered in R in one of two possible ways.

Periods larger than n are given by cycles of the form $I_1 \to \ldots \to I_{n-1} \to I_1 \to \ldots \to I_1$.

The smaller even periods are given by cycles of the form $I_{n-1} \to I_{n-2} \to I_{n-1}$, $I_{n-1} \to I_{n-4} \to I_{n-3} \to I_{n-2} \to I_{n-1}$ and so on

Case (ii): n is even

For n even, *f* must have a periodic point of period 2.

We assure that some x_i's change sides under *f* and some do not ($I_{n-1} \to I_{n-2}$ and $I_{n-2} \to I_{n-1}$).

If this is not the case, then all x_i's must change sides and so $[x_{i+1}, x_n] \subset f[x_1, x_i]$ and $[x_1, x_i] \subset f[x_{i+1}, x_N]$.

But then, it produces a period 2 point in $[x_1, x_i]$.

Now, we will prove it for n = 2^m, where m$\geq$ 0.

For $l <$ m, consider k = 2^l.

Let g = $f^{k/2}$.

By assumption, g has a periodic point of period 2^{m-l+1}.

$\therefore$ g has a point of period 2.

$\Rightarrow$This point has period k = 2^l for *f*.

Next, we prove it for n = p. 2^m with p odd and m $\geq$ 0.

For $l >$ m, consider k = 2^l.q.

Let g = f^m.

We know that g has a periodic point of period p.

Since g has a point with period q.2^{l-m} for $l >$ m.

$\Rightarrow$This point has period k = q.2^l under *f*.

Thus, *f* has a point with period k = $2^{l-m}.2^m \Rightarrow$ k = 2^l.

$\therefore$ *f* has a periodic point of prime period k.

B. Converse Of Sarkovskii Theorem

For any number n, there is a function *f*: R$\to$R such that *f* has a periodic point of prime period n but has no periodic points of any prime period k preceding in the sarkovskii ordering.

PROOF:

Case (i):

Let n = 2k+1.

Let *f*: [1, n]$\to$[1, n] be a continuous function having cycles of period 2k+1 but not of 2k-1.

$$\text{Let } f(x) = \begin{Bmatrix} kx+1 \quad if\ 1 \leq x \leq 2, \\ 2k+3-x if\ 2 < x \leq k+1, \\ 3k+4-2x if k+1 < x \leq k+2, \\ 2k+2-x if k+2 < x \leq 2k+1 \end{Bmatrix}$$

Since *f* is a continuous function.

$\Rightarrow$ The cycle of period 2k+1 is
1$\to$k+1$\to$k+2$\to$k$\to$k+3$\to$k-1$\to$…..$\to$2k+1$\to$1 where k+1+i$\to$ k+1+I $\to$ k+1+ (i-1)$\to$….. holds for the integer i such that 1$\leq$ i $\leq$ k.

*Claim:*To compute f^{2k-1}[1, 2].

f maps the following closed intervals as [1, 2]$\to$ [k+1, 2k+1]$\to$[1, k+2]$\to$[k, 2k+1]$\to$….$\to$[1, 2k]$\to$[2, 2k+1] where [1, k+i]$\to$ [k+2-i, 2k+1]$\to$[1, k+1+i] holds for 2$\leq$ i $\leq$ k.

It gives that f^{2k-1}[1, 2] = [2, 2k+1].

The point x = 2, the intersection of the last two intervals is a part of the (2k+1)cycle, so it is not a fixed point of f^{2k-1} and this function has no other fixed points in [1, 2].

Similarly, none of the other intervals [1, 2k+1] of the form [i, i+1] has a fixed point of f^{2k-1} except for [k+1, k+2] such that [k+1, k+2]$\to$[k, k+2]$\to$[k,k+3]$\to$[k-1,k+3]$\to$…$\to$[1,2k+1] Where [k+1-i, k+1+i] $\to$ [k+1-i, k+1+ (i+1)] $\to$[k+1-(i+1), k+1+ (i+1)] holds for 1$\leq$ i $\leq$ k < 1.

It follows that f^{2k-1}[k+1, k+2] = [1, 2k+1].

$\Rightarrow f^{2k-1}$ must have a fixed point in that interval.

Now, we will show that point is unique and hence it is the fixed point of *f* which is not in any cycle.

Since *f* is monotonically decreasing over each of the intervals in the sequence of mappings.

Indeed, the composition of an odd number of monotonically decreasing function is monotonically decreasing, that the fixed point is unique.

Since f^{2k-1} crosses the line y = x only once as it goes through the interval [k+1, k+2].

$\therefore$ *f* has no cycles of period 2k-1.

*Case (ii)***:** n = p.2^k.

Let *f* be a continuous function of period p odd with cycle but no cycle of period p-2.

Let *g* be the function obtained by applying the double map k times to *f*.

$\Rightarrow$*g* has exactly a cycle of period n = 2^k.p but not of n = 2^k (p-2).

Otherwise,*f* would have had a cycle of period p-2.*Case (iii):* n = 2^k.

Let *f* (x) = 3-x mapping [1, 2] to itself.

Since all the points in that interval has prime period 2 except for the fixed point 3/2.

By applying the double map k-1 times to *f*, we get a desired function *g* as it generates a function with a point of period n = 2^k but none of n = 2^{k+1}.

Otherwise, *f* would have had a point of prime period 4.

Case (iv): n = 3.2^k.

Let f (x) be defined from [1, 3] to itself by *f* (1) = 2, *f*(2) = 3 and *f* (3) = 1 connecting by lines on [1,2] and [2,3].

Clearly, it has a point of prime period 3.

$\Rightarrow$ For k = 0, *f* satisfies the case n = 3.

By applying the double map k times to *f*, we get the desired function *g* as it has a point of period n = 3.2^k but none of n = p.2^{k-1}.

[Since there are no integers q such that 2^kq= 2^{k-1}.p].

$\therefore$ *f* has no periodic points of prime period k.

IV. APPLICATION

A. Detecting Stress in the Focusing Control Mechanism of the Short-SightedEye

It has been found that accommodative micro fluctuations are chaotic in nature. Thus, chaos is a potential factor in the accommodation control mechanism. A marker of chaos is sensitivity to initial conditions i.e., butterfly effect. If there is a miniscule change in the initial conditions, the resulting evolution of the system over time will be very different.

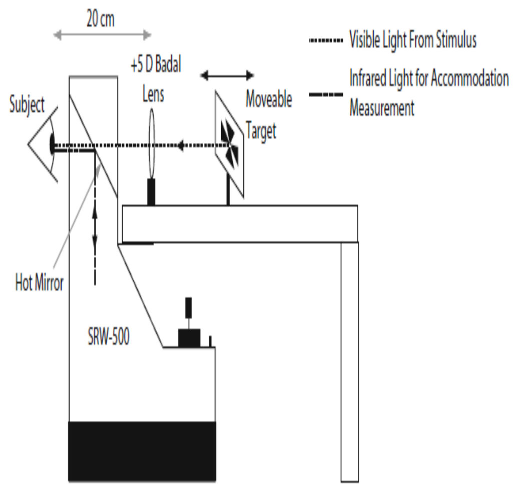

Figure 7

The target is viewed through a hot mirror which transmits visible light. The infrared light used for measuring the accommodation fluctuations are reflected into the eye through the hot mirror.

Accommodation micro fluctuations were measured using the SRW-5000 autorefractor which was modified to allow for continuous recording of accommodation at a sampling rate of 22Hz, while retaining the ability to measure static refractive error.

A marker of chaos, i.e., sensitive dependence on initial conditions, is a positive largest Lyapunov Exponent. The Lyapunov Exponent describes how the distance between nearby trajectories in phase space changes exponentially over time.

B. Phase Space Reconstruction

Phase space is effectively a plot in which each axis represents a variable. When recording a single variable over time, a multidimensional plot is formed using lagged phase space. Essentially, the time course signal is broken down into overlapping segments separated by an embedding lag τ. Each segment is a subseries representing the values for that axis.

The number of axis is governed by the embedding dimension *d*. The resulting phase space plot represents the dynamics of the system as it would be had each of the variables been known or measured. Using this, a point $p(t)$ in a phase space with d dimension is given by

$p(t)=[x(t),x(t+\tau),\ldots,x(t+(d-1)\tau)]$ (1)

where the embedding lag $\tau = t_0 + i\Delta t$, with Δt being the time between frames. The number of data points, $n_{ss.}$ per subseries is given by

$n_{ss} = N - (d-1) \times i(2)$

Where N is the total number of data points in the original time record and i is the embedding lag in units of data points.

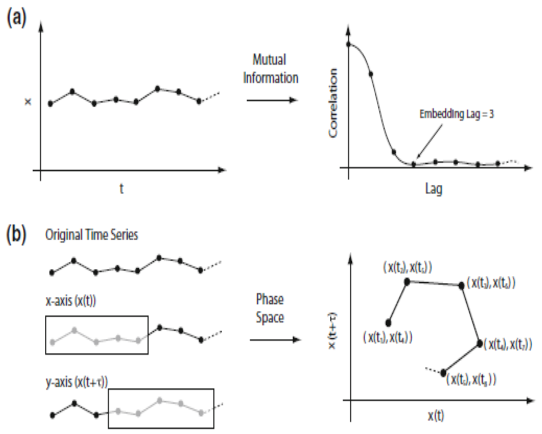

Figure 8: Principle of obtaining a multi-dimensional phase space

We determine the embedding lag using the first minimum of the mutual information. A schematic of obtaining a multi-dimensional phase space plot based on the determined lag. In below diagram, the embedding dimension is two. The mutual information is in effect a correlation between the signal and a delayed version of itself.

For two series x and y, the mutual information is given by

$$I_{XY} = \sum_{x_i,y_j} P_{XY}(x_i,y_j) log_2 \left(\frac{P_{XY}(x_i,y_j)}{P_X(x_i)P_Y(y_j)}\right) \quad (3)$$

Where P_{XY} is the joint probability and $P_X \& P_Y$ are the marginal probabilities.

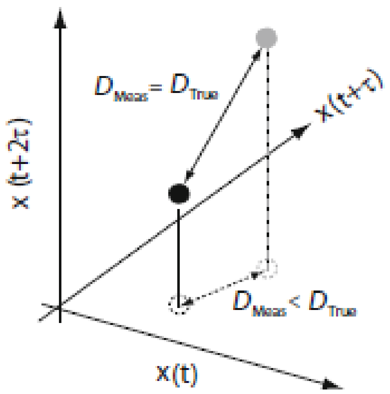

Figure 9: Obtaining the correct embedding dimension

To determine τ, the two time series are the original time series and the larged time series.

Hence $$I(\tau) = \sum_{x(t),x(t+\tau)} P(x(t),x(t+\tau)) log_2 \left(\frac{P(x(t),x(t+\tau))}{P(x(t)P(x(t+\tau))}\right) \quad (4)$$

$I(\tau)$ was calculated for each accommodation record for a lag of i = 1-50 data points.

To determine the correct dimension, the phase space plot is obtained for a number of dimensions. The separation of neighbouring points is then calculated for each dimension.

If the dimension is too low, the points are artificially too close and referred to as false nearest neighbours. The correct dimension is the one in which further increases in the dimension do not change the distance between the majority of data points and so the number of false nearest neighbor falls below a given threshold.

A data point located in d-dimensional space is given by

$$p(t) = [x(t), x(t+\tau), \ldots, x(t+(d-1)\tau)] \quad (5)$$

and its nearest neighbor is given by

$$P_{NN}(t_{NN}) = [x(t_{NN}), x((t_{NN}+\tau), \ldots, x(t_{NN}+(d-1)\tau)] \quad (6)$$

The distance between the points is

$$R_d^2(\mathrm{p}, P_{NN}) = \sum_{k=1}^{d}[x(t + (k-1)\tau) - x(t_{NN} + (k-1)\tau]^2. \quad (7)$$

The separation of points when the dimension increases by one is given by

$$R_{d+1}^2(\mathrm{p}, P_{NN}) = R_d^2(\mathrm{p}, P_{NN}) + [x(t + d\tau) - x(t_{NN} + d\tau)]^2 \quad (8)$$

Hence, the change in distance can be calculated as

$$\sqrt{\frac{R_{d+1}^2(\mathrm{p},P_{NN}) - R_d^2(\mathrm{p},P_{NN})}{R_d^2(\mathrm{p},P_{NN})}} = \frac{|x(t+d\tau) - x(t_{NN}+d\tau)|}{R_d(\mathrm{p},P_{NN})} \quad (9)$$

A point is considered as a false nearest neighbor if two conditions are satisfied:

$$\sqrt{\frac{R_{d+1}^2(\mathrm{p},P_{NN}) - R_d^2(\mathrm{p},P_{NN})}{R_d^2(\mathrm{p},P_{NN})}} > R_{Tol} \quad (10)$$

And

$$\frac{R_{d+1}}{R_A} > A_{Tol}. \quad (11)$$

Where R_A is the standard deviation of the time series.

Condition two prevents the number of false nearest neighbours rising again as d increases beyond the appropriate dimension, owing to the fact that the nearest neighbor of a point may not necessarily be the one closed to it. For a given dimension, the nearest neighbor of each point is determined equation (10) & (11) are evaluated for each pair of points.

The correct embedding dimension is the value in which the number of false nearest neighbor falls below a given threshold.

CONCLUSION

Chaos exists everywhere in nature, from the dynamics of the weather to the heartbeat. It is almost applicable everywhere in the field. Everything in the universe is under the control of chaos or the product of chaos. Irregularity leads to the complex system. It is sensitive with the initial conditions which makes the system fairly unpredictable. It never repeats but it has some order. The Chaos theory gives us a new way of measurements and scales. Through the theorems, we came to know that it deduces the existence of cycles of certain periods from the existence of cycles of a different period.

Also, we found that the dynamics of the micro fluctuations in accommodation remain chaotic irrespective of accommodative demand and refractive error. We came to know how to determine the chaos in fluctuations of the power of the lens changes with refractive error.

Chaos theory applied to micro fluctuations in accommodation tells us that differences are not detected using traditional analysis method. Thus by applying Sarkovskii's theorem concept of Chaos theory helps us to know that the infinite times of oscillation of a points contracts and converges to the inner element which helps us to study of patterns.

References

[1] Boieng, G.(2016). "Visual Analysis of Nonlinear Dynamical Systems: Chaos, Fractals,Self-Similarity and the Limits of Prediction".System.4(4):37.doi:10.3390/systems4040037. Retrieved2016-12-02.

[2] Carlo F.Barenghi. (2010). "Introduction to chaos: theoretical and numerical methods.

[3] Hristu-Varsakelis, D.; Kyrtsou, C. (2008). "Evidence for nonlinear asymmetric causality in US inflation, metal and stock returns". Discrete Dynamics in Nature and Society. 2008:1-7.doi:10.1155/2008/138547.138547.

[4] Kyrtsou, C.; M. Terraza, (2003). "Is it possible to study chaotic and ARCH behavior jointly? Application of a noisy Mackey-Glass equation with heteroskedastic errors to the Paris Stock Exchange returns series". Computational Economics.21(3):257-276.doi:10.1023/A:1023939610962.

[5] Dilao, R.; Dpmingos,T.(2001). "Periodic and Quasi-Periodic behavior in Resource Dependent Age Structured Population Models". Bulletin of Mathematical Biology. 63(2):207-230.doi:10.1006/bulm.2000.0213.PMID 11276524.

[6] Safonov, Leonid A.; Tomer, Elad; Strygin, Vadim V.; Ashkenazy, Yosef; Halvin, Shlomo(2002). "Multifractal chaotic attractors in a system of delay-differential equations modeling road traffic". Chaos: an Interdisciplinary Journal of Nonlinear Science. 12(4):1006.

[7] Bibcode: 2002chaos. . 12.1006S. Doi:10.1063/1.1507903.ISSN 1054-1500.

[8] Robert L. Devaney "An Introduction to Chaotic Dynamical Systems".

[9] Massimo Cencini, Fabio Cecconi.; Angelo Vulpiani. "Introduction to Dynamical Systems and Chaos". Series on advances in Statistical Mechanics- Vol 17. From Simple Models to Complex Systems.

[10] David Levy. "Chaos Theory and Strategy: Theory, Application and Managerial Implications".

[11] Karen M. Hampson.; Mathew P. Cufflin.; Edward A.H. Mallen. "Sensitivity of Chaos Measures in Detecting Stress in the Focusing Control Mechanism of the Short-Sighted Eye".

[12] Ott, Edward (2002). Chaos in Dynamical System. Cambridge University Press.

[13] Alligood, K.T.; Sauer, T.; Yorke, J.A. (1997). Chaos: an introduction to Dynamical Systems. Springer – Verlag. ISBN 0 - 387 – 94677 – 2.

[14] Lorenz, Edward.N. (1963). "Deterministic non-periodic flow". Journal of the Atmospheric Science. 20(2): 130-141. Bibcode: 1963JAtS…20..130L. doi:10.1175/1520-0469(1963)020<0130:DNF>2.0.CO;2.

[15] Fu,Z.; Heidel, J. (1997). "Nonchaotic behavior in three-dimensional quadratic systems". Nonlinearity. 10(5): 1289-1303. Bibcode: 1997Nonli..10.1289F. doi.10.1088/0951-7715/10/5/014.

[16] Sterman. J.D. (1989). "Deterministic Chaos in an experimental economic system", Journal of Economic Behaviour and Organization, 12(29), pp, 1-28.