Pseudo Regular and Pseudo Normal Topological Spaces

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Abstract: In this paper pseudo regular and pseudo normal topological spaces have been defined and their properties have been studied. A number of important theorems regarding these spaces have been established.

Keywords: Pseudo regular spaces, pseudo normal spaces, compact set, Hausdorff spaces, equivalence relation, projection mapping.

Mathematics Subject Classification: 54D10, 54D15, 54A05, 54C08.

I. INTRODUCTION

Regular and normal topological spaces have been generalized in various ways. P-regular, p-normal, β-normal and γ-normal spaces ([2], [3], [5], [7], [10]) are several examples of some of these. Here we have introduced pseudo regular and pseudo normal spaces and studied their important properties. Many results have been proved about the**se** spaces. We have also established characterizations of such spaces. Parallel study of further generalizations using preopen and semi open sets etc. are intended to be done in near future. We have used the terminology and definitions of text book of (S. Majumdar and N. Akhter, Munkres, Dugundji, Simmons, Hoking-Yong and Kellely). Unless otherwise stated, every compact set considered in this paper will be at least two elements.

II. PRELIMINARIES

We start with the definitions of almost γ-normal, almost p-normal, almost β-normal, γ-normal, p-normal, β-normal spaces.

A subset A of a topological space X is said to be **regular open** (resp. **regular closed**) if A=int(cl(A)) (resp. cl(int(A)),**preopen** (briefly **p-open**) if A $\subseteq$ int(cl(A)), β **-open** if A $\subseteq$ cl(int(cl(A))), γ **-open** if A $\subseteq$ cl(int(A)) $\cup$ int(cl(A)).

Definition 2.1: A topological spaces X is said to be **almost** γ **-normal** (resp. **almost p-normal**, **almost** β **-normal** [2]) if for any two disjoint closed subsets A and B of X, one of which is regularly closed, there exist disjoint γ-open (resp. p-open, β-open) sets U and V of X such that A $\subseteq$ U and B $\subseteq$ V.

Definition 2.2: A topological spaces X is said to be γ **-normal** (resp. **p-normal**, β **-normal** [2]) if for every pair of disjoint closed subsets A and B of X, there exist disjoint γ-open (resp. p-open, β-open) sets U and V of X such that A $\subseteq$ U and B $\subseteq$ V.

Their inter-relationships are also mentioned below:

normal $\Rightarrow$ almost normal
$\Downarrow$ $\Downarrow$
p-normality $\Rightarrow$ almost p-normality
$\Downarrow$ $\Downarrow$
γ-normality $\Rightarrow$ almost γ-normality
$\Downarrow$ $\Downarrow$
β-normality $\Rightarrow$ almost β-normality

We now define pseudo regular spaces and proceed to study them.

III. PSEUDO REGULAR SPACES

Definition 3.1: A topological space X is pseudo regular if every compact subset K of X and every $x \in X$ with $x \notin K$ can be separated by disjoint open sets.

Ex 3.1: Let K be a compact subset of R^n and let $x \in \mathrm{R}^n$ such that $x \notin K$. Since R^n is T_1, $\{\mathrm{x}\}$ is closed and since R^n is normal and K is closed (by Heine Borel Theorem), $\{\mathrm{x}\}$ and K can be separated by disjoint open sets. Thus R^n is **pseudo regular**.

Ex 3.2: Let X = {a,b,c,d} and $T = \{X, \phi, \{a,b\}, \{c,d\}\}$. Then (X,$T$) is **regular** but not **pseudo regular.**

For, {a,c}is compact, b $\notin \{a,c\}$, but {a,c} and b cannot be separated by disjoint open sets.

Ex 3.3: Let X = $\Re$, T be the topology generated by $\Im_0 \cup \zeta$ where $\Im_0$ is the usual topology on $\Re$ and $\zeta = \{\{x\} \mid x \in \Re - Q\}$. Then Q is closed, since $\Re$ -Q is open. Q cannot be separated from an irrational point since the only open set which contains Q is $\Re$. Therefore X is not regular. The compact sets in X are the closed and bounded subsets of $\Re$, i.e., finite unions of closed intervals, e.g., $[a_1,b_1] \cup \ldots\ldots\ldots\ldots \cup [a_n, b_n]$. Let K be a compact subset of X and let $x \notin K$. So let K be given by $K=[a_1,b_1] \cup \ldots\ldots\ldots\ldots \cup [a_n, b_n]$. Let $\delta_i = d(x,[a_i,b_i])$, the distance of x from $[a_i,b_i]$ and let $\delta = \frac{1}{2}\min\{\delta_1,\ldots\ldots\ldots,\delta_n\}$. Then G= $(a_1-\delta,b_1+\delta) \cup \ldots\ldots\ldots \cup (a_n-\delta,b_n+\delta)$ and H= $(x-\delta,x+\delta)$, separate K and x. Thus X is **pseudo regular** but **not regular**.

Theorem 3.1: Every pseudo regular compact space is regular.

Proof: Let X be compact and pseudo regular. Let K be a closed subset of X and let $x \in X$ with $x \notin K$ .Since X is compact, K is compact. Again, since X is pseudo regular, $\exists$ disjoint open sets G and H such that $x \in G$ and $K \subseteq H$. Therefore X is regular.

Theorem 3.2: Every regular T_2 -space is pseudo regular.

Proof: Let X be a regular T_2 -space. Let K be a compact subset of X and $x \in X$, $x \notin K$. Since X is T_2, K is closed. Now, since X is regular, $\exists$ disjoint open sets G and H such that $x \in G$ and $K \subseteq H$. Therefore X is pseudo regular.

Theorem 3.3: A topological space X is pseudo regular iff $\forall$ $x \in X$ and any compact set K not containing x, $\exists$ a open set H of X such that $x \in H \subseteq \overline{H} \subseteq K^c$.

Proof: Let X be pseudo regular and let K be compact in X. Let $x \notin K$ i.e; $x \in K^c$. Since X is pseudo regular, $\exists$ open sets U, V such that $x \in U, K \subseteq V$ and $U \cap V = \phi$. Then $U \subseteq V^c \subseteq K^c$. So $\overline{U} \subseteq \overline{V^c} = V^c \subseteq K^c$. Writing U=H we have $x \in H \subseteq \overline{H} \subseteq K^c$.

Now let $\forall x \in X$ and any compact set K not containing x, $\exists$ open set H such that $x \in H \subseteq \overline{H} \subseteq K^c$. Since K is a compact set and $x \notin K$. Then $x \in K^c$. According to condition, $\exists$ open set H such that $x \in H \subseteq \overline{H} \subseteq K^c$. Let $\overline{H}^c$ = G. Then G is open, $K \subseteq G$ and $G \cap H = \phi$. Thus X is pseudo regular.

Theorem 3.4: Product space X of any non-empty collection of $\{X_i\}$ is pseudo regular iff each X_i is pseudo regular.

Proof: Let $\{X_i\}$ be a non-empty collection of pseudo regular space and X = ΠX_i. We show that X is pseudo regular space. Let K be a compact set not containing a point $x \in X$. Let $K_i = \Pi_i(K)$, $x_i \notin K_i$. Since the projection maps are continuous $\Pi_i(K) = K_i$ is a compact subset of X_i. Since $x \notin K$, $\exists$ i_0 such that $x_{i_0} \notin K_{i_0}$. Since X_{i_0} is pseudo regular, $\exists$ disjoint open sets G_{i_0}, H_{i_0} in X_i such that $x_{i_0} \in H_{i_0}$, $K_{i_0} \subseteq G_{i_0}$. For each $i \neq i_0$, let G_i, H_i be open sets such that $x_i \in H_i, K_i \subseteq G_i$. Let G= $\Pi_i G_i$ and H= $\Pi_i H_i$. Then $G \cap H = \phi$, since $G_{i_0} \cap H_{i_0} = \phi$. Now, $K \subseteq G$, $x \in H$. Hence X is pseudo regular.

Conversely, if X is pseudo regular, then we show that for each i, X_i is pseudo regular. For each i, let K_i be a compact subset of X_i and $x_i \in X_i$ but $x_i \notin K_i$. Let $K = \Pi_i K_i$ and x= $\{x_i\}$, $x \in X$ but $x \notin K$. Then K is compact by Tychonoff Theorem. Since X is pseudo regular, $\exists$ disjoint open sets G and H such that $x \in G$ and $K \subseteq H$ and G= $\Pi_i G_i$, H= $\Pi_i H_i$, G_i, H_i are open sets in X_i such that $x_i \in H_i, K_i \subseteq G_i$ and $G_i \cap H_i = \phi$. Therefore X_i is pseudo regular.

Theorem 3.5: Any subspace of a pseudo regular space is pseudo regular.

Proof: Let X be a pseudo regular space and $Y \subseteq X$. Let $y \in Y$ and B is a compact subset of Y such that $y \notin B$. Since B is compact in Y, so B is compact in X. Since X is pseudo regular, $\exists$ disjoint open sets G and H of X such that $y \in G$ and $B \subseteq H$. Let U= $G \cap Y$ and V= $H \cap Y$. Then U and V are disjoint open sets of Y where $y \in U$ and $B \subseteq V$. Hence Y is pseudo regular.

Corollary 3.1: Let X be a topological space and A, B are two pseudo regular subspace of X. Then $A \cap B$ is pseudo regular.

Proof: $A \cap B$ being a subspace of both A and B, $A \cap B$ is pseudo regular by the above theorem.

Theorem 3.6: Let X be a pseudo regular space and R is an equivalence relation of X. Then R is a closed subset of $X \times X$.

Proof: We shall prove that R^c is open. So, let $(x, y) \in R^c$. It is sufficient to show that there exists two open sets G and H of X such that $x \in G$ and $y \in H$ and $G \times H \subseteq R^c$. Let $P: X \to \frac{X}{R}$ be the projection map. Since $(x, y) \in R^c$, $p(x) \neq p(y)$ i.e; $x \notin p^{-1}(p(y))$. Again, since $\{y\}$ is compact and P is a continuous mapping, p(y) is compact. Also, let $\{G_i\}$ be an open cover of $p^{-1}(p(y))$ in X, and let $\overline{G_i} = p(G_i)$. Then $\{\overline{G_i}\}$ is an open cover of p(y) in $\frac{X}{R}$. Since p(y) is a singleton element in $\frac{X}{R}$, $\exists$ $\overline{G_{i_0}}$ s. t. $p(y) \in \overline{G_{i_0}}$ in $\frac{X}{R}$. Then by the definition of the topology in $\frac{X}{R}$ and the nature of the map p, (i) G_{i_0} is open in X, (ii) $G_{i_0} = p^{-1}(\overline{G_{i_0}})$ and (iii) $p^{-1}(p(y)) \subseteq G_{i_0}$ in X. Hence $p^{-1}(p(y))$ is compact in X. So by the pseudo regularity of X $\exists$ disjoint open sets G and H in X such that $x \in G$ and $p^{-1}(p(y)) \subseteq H$. Hence $y \in p^{-1}(p(y)) \subseteq H$ i.e; $y \in H$. Since $G \cap H = \phi$, $p(G) \cap p(H) = \phi$. Therefore $G \times H \subseteq R^c$ and so $(x, y) \in G \times H \subseteq R^c$.

Corollary 3.2: Let X be a pseudo regular space and R is an equivalence relation of X. Then $\frac{X}{R}$ is Hausdorff.

Proof: Let $\bar{x}$ and $\bar{y}$ be two distinct points of $\frac{X}{R}$. Then $\bar{x} = p(x)$ and $\bar{y} = p(y)$ for some $x, y \in X$ such that $x \neq y$ and $(x, y) \in R^c$. By the proof of the above theorem, $\exists$ disjoint open sets $\overline{G}$ and $\overline{H}$ in $\frac{X}{R}$ such that $\bar{x} \in \overline{G}$ and $\bar{y} \in \overline{H}$. Thus $\frac{X}{R}$ is Hausdorff. [$\overline{G}$ =p(G) and $\overline{H}$ =p(H) of the above theorem].

Theorem 3.7: Every locally compact Hausdorff space is pseudo regular.

Proof: Let X be a locally compact Hausdorff space. Then there exists one point compactification X_∞ of X and X_∞ is Hausdorff and compact. According to above Remark X_∞ is pseudo regular. Again, according to Theorem 3.5, as a subspace of X_∞, X is pseudo regular.

We now define pseudo normal spaces and proceed to study them.

IV. PSEUDO NORMAL SPACES

Definition 4.1: A topological space X is pseudo normal if each pair of disjoint compact subsets of X can be separated by disjoint open sets.

Ex 4.1: Since R^n is normal and every compact subset of R^n is closed, R^n is pseudo normal.

Ex 4.2: Let X = {a,b,c,d} and $T = \{X, \phi, \{a,b\}, \{c,d\}\}$. Then (X,T) is a **normal space**. Here {a,c}and {b,d}are two disjoint compact sets in X, but there do not exist disjoint open sets containing these compact sets. Therefore (X,T) is **not pseudo normal**.

Theorem 4.1: Every pseudo normal compact space is normal.

Proof: Let X be compact and pseudo normal. Let A, B be two disjoint closed subsets of X. Since X is compact, A and B are compact. Again, since X is pseudo normal, A and B can be separated by disjoint open sets. Therefore X is normal.

Theorem 4.2: Every normal T_2-space is pseudo normal.

Proof: Let X be T_2 and normal. Let A, B be two disjoint compact subsets of X. Since X is T_2, A, B are closed. Again, since X is normal, $\exists$ disjoint open sets G and H in X such that $A \subseteq G$ and $B \subseteq H$. Therefore X is pseudo normal.

Theorem 4.3: A topological space X is pseudo normal iff each pair of disjoint compact sets K_1 and K_2 $\exists$ open set U such that $K_1 \subset U \subset \overline{U} \subset K_2{}^c$.

Proof: Let X be a pseudo normal and K_1, K_2 be two compact subsets of X and $K_1 \cap K_2 = \phi$. Since X is pseudo normal, $\exists$ open sets U, V such that $K_1 \subset U$, $K_2 \subset V$ and $U \cap V = \phi$. Then $U \subset V^c \subset K_2{}^c$. So $\overline{U} \subset \overline{V^c} = V^c \subset K_2{}^c$. Hence we have $K_1 \subset U \subset \overline{U} \subset K_2{}^c$.

Conversely, suppose that for each pair K_1 and K_2 of disjoint compact subsets of X there exist an open set H of X such that $K_1 \subset H \subset \overline{H} \subset K_2^{\,c}$. We shall show that X is pseudo normal. Here $K_1 \subset H$ and $K_2 \subset \overline{H}^{\,c}$. Let $\overline{H}^{\,c} = G$. Then G is open , $K_2 \subset G$ and $G \cap H = \phi$. [For $x \in H \cap \overline{H}^{\,c} \Rightarrow x \in H$ and $x \in \overline{H}^{\,c}$. But $x \in H \Rightarrow x \in \overline{H}$. So $x \notin \overline{H}^{\,c}$ which is a contradiction, so $G \cap H = \phi$]

Theorem 4.4: Every open image of a pseudo normal space is pseudo normal.

Proof: Let X be a pseudo normal space and Y a topological space and let $f : X \to Y$ be an open and onto mapping. Let K_1 and K_2 be two disjoint compact subsets in Y. Then $f^{-1}(K_1)$ and $f^{-1}(K_2)$ are compact in X. Since X is pseudo normal $\exists$ open subsets U and V of X such that $f^{-1}(K_1) \subset U$ and $f^{-1}(K_2) \subset V$ and $U \cap V = \phi$. Again, since f is open f(U) and f(V) are open in Y and $K_1 \subset ff^{-1}(K_1) \subset f(U)$, $K_2 \subset ff^{-1}(K_2) \subset f(V)$ and $f(U) \cap f(V) = \phi$. Hence Y is pseudo normal.

Corollary 4.1: Every quotient space of a pseudo normal space is pseudo normal.

Proof: Let X be a pseudo normal space and R is an equivalence relation on X. Since the projection map P:X $\to \frac{X}{R}$ is open and onto, the corollary then follows from the above theorem.

Theorem 4.5: Every subspace of a pseudo normal space is pseudo normal.

Proof: Let X be a pseudo normal space and $Y \subseteq X$. Let K_1 and K_2 be two disjoint compact subsets in Y. Since K_1 and K_2 are compact in Y, these are compact in X too. Since X is pseudo normal, there exist disjoint open sets U and W such that $K_1 \subset U$ and $K_2 \subset W$. Then $U \cap Y$ and $W \cap Y$ are disjoint open sets in Y with property that $K_1 \subset U \cap Y$ and $K_2 \subset W \cap Y$. Hence Y is pseudo normal.

Comment 4.1: A continuous image of a pseudo regular (pseudo normal) space need not be pseudo regular (pseudo normal).

For if (X, T_1) is a pseudo regular (a pseudo normal) space and (X, T_2) a space with the indiscrete topology, then the identity map $1_x : X \to X$ is continuous and onto. But (X, T_2) is not pseudo regular (pseudo normal).

Theorem 4.6: Each compact Hausdorff space is pseudo normal.

Proof: Let X be compact Hausdorff space and A,B be two disjoint compact subsets of X. Let $x \in A$ and $y \in B$. Then $x \neq y$. Since X is Hausdorff there exist disjoint open sets G_y and H_y such that $x \in G_y$ and $y \in G_y$. Obviously $\{H_y : y \in B\}$ is an open cover of B.

Since B is compact, so there exist finite subcover $H_{y_1}, H_{y_2}, \ldots\ldots, H_{y_m}$ of B. Let $H_x = H_{y_1} \cup H_{y_2} \cup \ldots\ldots \cup H_{y_m}$ and $G_x = G_{y_1} \cap G_{y_2} \cap \ldots\ldots \cap G_{y_m}$. Then $B \subseteq H_x$, $x \in G_x$ and $H_x \cap G_x = \phi$ i.e; X is pseudo regular. So for each $x \in A$ there exist two disjoint open sets G_x and H_x of X such that $x \in G_x$ and $B \subseteq H_x$. Hence $\{G_x : x \in A\}$ is a open cover of A. Since A is compact, so there exist finite subcover $G_{x_1}, G_{x_2}, \ldots\ldots, G_{x_n}$ of this cover A. Let $G = G_{x_1} \cup G_{x_2} \cup \ldots\ldots \cup G_{x_n}$ and $H = H_{x_1} \cap H_{x_2} \cap \ldots\ldots \cap H_{x_n}$. Then G, H are open sets of X and $A \subseteq G$, $B \subseteq H$ and $G \cap H = \phi$.

Remark 4.1: It follows from the above proof that every compact Hausdorff space is pseudo regular.

Note: Since X is Hausdorff and every compact set is closed. Therefore every closed set F which is compact can be separated by open sets from a point not containing F.

Theorem 4.7: Let X be a topological space such that for every compact subset K of X, X-K contains at least two elements, if each X is pseudo normal then X is pseudo regular.

Proof: Let X be a pseudo normal space. Let K be a compact subset of X and let $x \in X$ such that $x \notin K$. Then there exists y such that y $\notin K$ and y $\neq$ x . Then {x,y}being finite with two elements, is a compact subset of X such that {x,y} $\cap K = \phi$. Since X is pseudo normal, $\exists$ open sets G and H with{x,y} $\subseteq G$, $K \subseteq H$, $G \cap H = \phi$. Since x $\in G$, $K \subseteq H$, $G \cap H = \phi$, hence X is pseudo regular.

Ex 4.3: Let X = {a,b,c} and $T = \{X, \phi, \{a\}, \{b,c\}\}$. Then (X,T) is a **pseudo normal** space. Again, since there do not exist two disjoint compact sets in X, it is **not pseudo regular**. For, {a,b} compact $c \notin \{a,b\}$, but {a,b} and c cannot be separated by disjoint compact sets in X.

Theorem 4.8: Every metric space is pseudo regular and pseudo normal.

Proof: Since every metric space is Hausdorff, every compact set is closed. Again since every metric space is regular, normal, therefore it is pseudo regular and pseudo normal.

V. ALMOST PSEUDO REGULAR SPACES AND ALMOST PSEUDO NORMAL SPACES

Here we consider two classes of topological spaces one of which lies between the class of Hausdorff spaces and the class of pseudo regular spaces, while the other lies between the class of Hausdorff spaces and the class of pseudo normal spaces.

Definition 5.1: Atopological space X is said to be almost pseudo regular if for every finite set A with at least two elements and for every $x \notin A$, $\exists$ disjoint open sets G and H such that $A \subseteq G$ and $x \in H$.

Ex 5.1: Let X= Q, r, $s \in Q$, r<s , and let $V_{r,s} = \{q \in Q \mid r < q < s\}$. Let T, the topology generated by $\{X, \phi, \{V_{r,s}\} \mid r, s \in Q, r < s\}$ .Let A={q_1,…….,q_n}, $n \geq 2$, and let $q \in Q$, $q \notin A$.Suppose $q_1 < q_2 < \ldots < q_n$. If $q < q_1$, let $\delta = q_1 - q$. Then $q \in V_{q-\delta/2,\, q+\delta/2}$ and $A \subseteq V_{q_1-\delta/2,\, q_n+\delta/2}$. If $q > q_n$, we construct the required open sets similarly.

If $q_i < q < q_{i+1}$, for some i,$1 \leq i < n$, let $\delta = \min(q - q_i, q_{i+1} - q)$. Then $q \in (q - \delta/2, q + \delta/2)$=V, say $q_1, \ldots, q_i \in (q_1 - \delta/2, q_2 + \delta/2) = V_1$ and $q_{i+1}, \ldots, q_n \in (q_{i+1} - \delta/2, q_n + \delta/2) = V_2$, say. Then $q \in V$, $A \subseteq V_1 \cup V_2$ and $V \cap (V_1 \cup V_2) = \phi$.

Definition 5.2: Atopological space X is said to be almost pseudo normal if for every two finite disjoint sets A and B each with at least two elements , $\exists$ disjoint open sets G and H such that $A \subseteq G$ and $B \subseteq H$.

Ex 5.2: It can be shown that the above Ex 5.1 is almost pseudo normal too. Many of the properties of the pseudo regular and pseudo normal spaces are expected to hold but we are not proving these here. We will follow up these in near future.

Acknowledgement

I express my gratitude to almighty God who creates and patience me to complete the work of this article.I would like to express my sincerest appreciation to my reverend supervisors**Professor Dr. Nasima Akhter andProfessor Dr. Subrata Majumdar,** Department of Mathematics, University of Rajshahi, Rajshahi-6205, Bangladesh, for their supervision, guidance, valuable suggestions, scholastic criticism, constant encouragement and helpful discussion throughout the work, I shall remain grateful to them.

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