Consequences of Unique Fixed points of λ - Generalized Contraction Selfmapings in D*-Metric Space

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Abstract: The main aim of this paper is to prove the consequences of fixed point on λ-generalized contraction of self mapping functions on D*-metric space.

Key Words: *D*-Metric Space, K-Contraction,* λ *-Generalized Contraction,*

I. INTRODUCTION

Huang and Zhang [11] generalized the notion of metric spaces, replacing the real numbers by an ordered Banach space and defined cone metric spaces. They have proved Banach contraction mapping theorem and some other fixed point theorems of contractive type mappings in cone metric spaces. Subsequently, Rezapour and Hamlbarani [17], Ilic and Rakocevic [9], contributed some fixed point theorems for contractive type mappings in cone metric spaces.

Gahler [7, 8] introduced the notions of 2-metric space and Dhage [5, 6] defned D-metric spaces as a generalization of metric spaces. In 2003, Zead Mustafa and Brailey Sims[13] introduced a new structure of generalized metric spaces, which are called G-metric spaces. Recently Aage and Salunke[2] generalized G -metric space by replacing R by real Banach space in G -metric spaces. In 2007 Shaban Sedghi et al [18] modify the D-metric space and defined D §-metric spaces. Now in this paper I Generalized D-metric spaces by introducing generalized D -metric space by replacing R by a real Banach space in D-metric spaces.

II. PRELIMINARY NOTES

Recall that a selfmap f of a D*-metric space (X, D*) is called a ***contraction***, if there is a q with $0 \le q < 1$ such that

(2.1) $$D^*(fx, fy, fy) \le q.D^*(x, y, y) \text{ for all } x, y \in X$$

In a different way R. Kannan [9] has defined a contraction for metric spaces which we shall call a K-contraction. Analogously we define the K-contractions for D*-metric spaces as follows:

(2.2) Definition: A selfmap f of a D*-metric space (X, D*) is called a ***K-contraction***, if there is a q with $0 \le q < \frac{1}{2}$ such that

(2.3) $$D^*(fx, fy, fz) \le q.\max\{D^*(x, fx, fx) + D^*(y, fy, fy)\} \text{ for all } x, y \in X$$

The notions of contraction and K-contraction are independent. In this thesis we define a special type of contractions called λ - generalized contractions for D*-metric spaces as follows:

(2.4) Definition: A selfmap f of a D*-metric space (X, D*) is called a λ ***-generalized contraction***, if for every $x, y \in X$, there exist non-negative numbers q, r, s and t (all depending on x and y) such that

(2.5) $$\sup_{x, y \in X} \{q + r + s + 2t\} = \lambda < 1 \text{ and}$$

(2.6) $$D^*(fx, fy, fz) \le q.D^*(x, y, y) + r.D^*(x, fx, fx) + s.D^*(y, fy, fy)$$
$$t.\{D^*(x, fy, fy) + D^*(y, fx, fx)\}$$

for all $x, y \in X$

As already noted in the Remark, every contraction and every K-contraction is a λ -generalized contraction. However the following examples show that there are some λ -generalized contraction f on a D*-metric spaces (X, D*) which are not contractions and /or K-contractions. The following is an example of a λ -generalized contraction which is not a contraction.

III. MAIN RESULT

Theorem: Suppose f is a selfmap of a D*-metric space (X, D*) and X is f-orbitally complete. If there is a positive integer k such that f^k is a λ -generalized contraction, then it has a unique fixed point $u \in X$. In fact,

(3.1) $u = \lim_{n \to \infty} f^n x$ for any $x \in X$

and

(3.2) $D^*(f^n x, u, u) \le \lambda^{n/k} . \rho(x, fx, fx)$ for all $x \in X$ and $n \ge 1$,

where $\rho(x, fx, fx) = \max\{\lambda^{-1} D^*(f^r x, f^{r+k} x, f^{r+k} x) : r = 0, 1, 2, \ldots, k-1\}$.

Proof: Suppose that f^k is a λ-generalized contraction of a f-orbitally complete D*-metric space (X, D*). By Theorem 3.1, f^k has unique fixed point. Let u be the fixed point of f^k. Then we claim that fu is also a fixed point of f^k.

In fact,

$$f^k(fu) = f^{k+1}u = f(f^k u) = fu$$

By the uniqueness of fixed point of f^k, it follows that $fu = u$, showing that u is a fixed point of f.

To prove the uniqueness of fixed point of f, let $v \in X$ be such that $fv = v$. Then $f^k v = v$ and the fact that f^k is a λ-generalized contraction, imply

$$\begin{aligned} D^*(u,v,v) &= D^*(f^k u, f^k v, f^k v) \\ &= qD^*(u,v,v) + rD^*(u, f^k u, f^k u) + sD^*(v, f^k v, f^k v) \\ &\quad + t\{D^*(u, f^k v, f^k v) + D^*(v, f^k u, f^k u)\} \\ &= qD^*(u,v,v) + rD^*(u,u,u) + sD^*(v,v,v) \\ &\quad + t\{D^*(u,v,v) + D^*(v,u,u)\} \\ &= (q+2t)D^*(u,v,v) \\ &\le \lambda . D^*(u,v,v) \end{aligned}$$

which implies that $D^*(u,v,v) = 0$ and hence $u = v$, proving uniqueness.

To prove (2.3.2), let n be any integer, then by the division algorithm, $n = mk + j,\ 0 \le j < k,\ m \ge 0$ and $x \in X$, $f^n x = (f^k)^m f^j x$. Since f^k is a λ-generalized contraction, by (2.2.3) we have

$$\begin{aligned} D^*(f^n x, u, u) &= D^*((f^k)^m f^j x, u, u) \\ &\le \frac{\lambda^m}{1-\lambda} D^*(f^j x, f^k f^j x, f^k f^j x) \\ &\le \frac{\lambda^m}{1-\lambda} D^*(f^j x, f^{k+j} x, f^{k+j} x) \end{aligned}$$

$$D^*(f^n x, u, u) \le \frac{\lambda^m}{1-\lambda} \max\{D^*(f^i x, f^{i+j} x, f^{i+j} x) : i = 0,1,2, \ldots, k-1\}$$

$$\to 0 \text{ as } n \to \infty$$

Thus $u = \lim_{n \to \infty} f^n x$ for any $x \in X$.

To prove (2.3.3), let n be any positive integer, f^k is a λ-generalized contraction and $n = mk + j,\ 0 \le j < k,\ m \ge 0$ with $m = \left[\frac{n}{k}\right]$, from (3.2) we have

$$D*\left(f^n x, u, u\right) = D*\left(f^{mk} f^j x, u, u\right)$$

$$\le \lambda^m D*\left(f^j x, f^{k+j} x, f^{k+j} x\right)$$

$$= \left(\lambda^{1/k}\right)^{mk+j-j} D*\left(f^j x, f^{k+j} x, f^{k+j} x\right)$$

$$\le \left(\lambda^{1/k}\right)^{mk+j-k} D*\left(f^j x, f^{k+j} x, f^{k+j} x\right)$$

$$\le \left(\lambda^{1/k}\right)^{n} \lambda^{-1}.D*\left(f^j x, f^{k+j} x, f^{k+j} x\right)$$

Hence

$$D*\left(f^n x, u, u\right) \le \lambda^{n/k} \max\left\{\lambda^{-1} D*\left(f^i x, f^{i+k} x, f^{i+k} x\right) : i = 0, 1, 2, \ldots, k-1\right\},$$

proving the theorem.

(3.3) **Theorem:** Suppose x_0 is a point in the D*-metric space (X, D*) and B is a closed ball of radius r about x_0. Suppose $f : B \to X$ is a

λ-generalized contraction on B, X is f-orbitally complete and

(3.4) $$D*\left(x_0, fx_0, fx_0\right) \le (1-\lambda) r,$$

where $\lambda = \underset{x, y \in X}{Sup} \{q + r + s + 2t\} < 1$ in which the numbers q, r, s and t are as in the Definition of λ-generalized contraction.

Then f has a unique fixed point $u \in B$.

(3.5) $$u = \lim_{n \to \infty} f^n x_0$$

and

(3.6) $$D*(f^n x_0, u, u) \le \lambda^n . r$$

Proof: Since $\lambda < 1$, from (3.4) it follows that $fx_0 \in B$, considering $x_1 = fx_0,\ x_2 = fx_1 = f^2 x_0, \ldots, x_n = fx_{n-1} = f^n x_0, \ldots,$ we shall show by induction that, this sequence is contained in B.

Suppose $x_0, x_1, x_2, \ldots, x_m \in B$. Then for $n = 1, 2, 3, \ldots, m$ and for n=0 and p=m+1, we have

$$D^*(x_0, x_{m+1}, x_{m+1}) \le \frac{1}{1-\lambda} D^*(x_0, fx_0, fx_0)$$

and by (3.4) we have

$$D^*(x_0, x_{m+1}, x_{m+1}) \le \frac{1}{1-\lambda} D^*(x_0, fx_0, fx_0) \le \frac{1}{1-\lambda}(1-\lambda).r = r\ ,$$

showing $x_{m+1} \in B$. Hence the sequence $\{f^n x_0 : n = 0,1,2,3,...\}$ is contained in *B*. using the same procedure as in the Theorem 3.1, this sequence can be proved to be a Cauchy sequence and hence has a limit $u \in X$, which must be a fixed point of f. Since *B* is closed $u \in B$. This completes the proof of the theorem.

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