Coefficient Estimate for New Subclasses of Ma-Minda bi-Univalent Function

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Abstract—In this paper, we introduce and investigate new subclass $H^r_{\psi}(y, \alpha, \lambda, \psi)$ of Ma-Minda bi-univalent functions in the open unit disk $U = \{z: z \in C \text{ and } |z| < 1\}$. For functions belonging to this class, we obtain estimate for the initial coefficient $|a_3|$ and $|a_4|$. The results derived in this paper would generalize those in related works of several earlier authors.

Keywords—Bi-Univalent functions, Ma-Minda starlike and Ma-Minda convex function

I. INTRODUCTION

Let $C(k)$ denote the class of the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disc $U = \{z: z \in C \text{ and } |z| < 1\}$. Also, let $S$ denote the class of all functions in $C(k)$ which are univalent in $U$.

For each $f \in S$, the Koebe one-quarter theorem [3, p.31] states that the image of the open unit disk $U$ under $f$ contains a disk of radius $1/4$. Thus, every univalent function $f$ has an inverse $f^{-1}$, which is defined by

$$f^{-1}(f(z)) = z, \quad z \in U$$

and

$$f(f^{-1}(w)) = w, \quad |w| < r_0(f) \geq 1/4,$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots.$$ (2)

A function $f \in C(k)$ is said to be bi-univalent in $U$. Let $\sigma$ denote the class of bi-univalent functions defined in the open unit disk $U$. In 1967, Lewin [1] studied the class of bi-univalent functions and obtained bound $|a_3| \leq 1.51$ for every $f \in \sigma$. Subsequently, Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}f$ for $f \in \sigma$. Netanyahu [4], on the other hand, proved that $\max_{f \in \sigma} |a_2| = 4/3$. The coefficient estimate problem for Taylor-Maclaurin coefficient $|a_n|$ for $n \in N - \{1,2,3\}$ is presumably still an open problem. Some of the important and well investigated subclasses of the univalent function class $S$ include the class $S^\ast(\alpha)$ of starlike functions of order $\alpha$ in $U$ and the class $K(\alpha)$ of convex functions of order $\alpha$ in $U$ which are defined as

$$S^\ast(\beta) = \{f \in S: \text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \quad (0 \leq \beta < 1; z \in U)\}$$ (3)

and

$$K(\beta) = \{f \in C(k): \text{Re}\left(1 + \frac{zf'(z)}{f(z)}\right) > \beta \quad (0 \leq \beta < 1; z \in U)\}.$$ (4)

It readily follows from the definition (3) and (4) that

$$f \in K(\beta) \iff zf^{-1} \in S^\ast(\beta).$$

Earlier, Brannan and Taha [5] (see also [6]) introduced certain subclasses of bi-univalent function class $\sigma$, namely $S^\ast_\sigma(\alpha)$ and $K_\sigma(\alpha)$ of bi-starlike and bi-convex functions of order $\alpha$ ($0 < \alpha \leq 1$), respectively. Thus, following Brannan and Taha [5] (see also [6]), a function $f \in C(k)$ is in the class $S^\ast_\sigma(\alpha)$ and $K_\sigma(\alpha)$ of strongly bi-starlike and strongly bi-convex functions of order $\alpha$ ($0 < \alpha \leq 1$), if both $f(z)$ and $f'(z)$ are strongly starlike and strongly convex functions of order $\alpha$, corresponding (respectively) to the function classes $S^\ast(\alpha)$ and $K(\alpha)$, were introduced analogously. For each of the function classes $S^\ast_\sigma(\alpha)$ and $K_\sigma(\alpha)$, non sharp estimates were found on the first two Taylor-Maclaurin coefficient $|a_3|$ and $|a_4|$ (for details, see [5] [6]).

An analytic function $f(z)$ is subordinate to an analytic function $g(z)$, written $f(z) \prec g(z)$, if there exist Schwarz function $w$ in $C(k)$, with $w(0) = 0$ and $|w(z)| < 1$. Satisfying $f(z) = g(w(z))$. In particular, when $g$ is univalent, then the above subordination is equivalent to $f(0) = 0$ and $f(U) \subset g(U)$. Ma and Minda [7] unified various subclasses of starlike and convex functions for which either of the quantities $zf'(z)/f(z)$ or $1 + zf'(z)/f(z)$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic univalent function $\varphi$ with positive real part in the open unit disk $U$, which maps $U$ onto a region symmetric with respect to the real axis and starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) = 0 > 0$. Such a function has series expansion of the form

$$\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \ldots \ldots \ldots \ldots$$ ($B_1 > 0$).

The classes $S^\ast(\varphi)$ and $K(\varphi)$ of Ma-Minda starlike and Ma-Minda convex functions consists of function $f \in C(k)$ respectively characterized by $zf'(z)/f(z) \prec \varphi(z)$ or $1 + zf'(z)/f(z) \prec \varphi(z)$. A function $f(z)$ is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both $f(z)$ and $f^{-1}(z)$ are respectively Ma-Minda type starlike or convex. These classes are denoted...
\[ 1 + \frac{1}{f} \left( 1 - \alpha + 2\lambda \right) \left( f'(z) \right)^{1/2} + \left( \alpha - 3\lambda \right) \left( f'(z) \right)^{1/2} + \frac{\lambda \left[ 1 + \frac{z}{f'(z)} \left( f'(z) \right)^{3/2} - 1 \right]}{\varphi(z)} \] (6)

and

\[ 1 + \frac{1}{f} \left( 1 - \alpha + 2\lambda \right) \left( \frac{g(w)}{w} \right)^{1/2} + \left( \alpha - 3\lambda \right) \left( \frac{g(w)}{w} \right)^{1/2} + \frac{\lambda \left[ 1 + \frac{w}{g(w)} \left( g'(w) \right)^{3/2} - 1 \right]}{\varphi(w)} , \] (7)

where \( g(w) = f'(w) \).

It is interesting to note that, for suitable choices of \( \gamma, \alpha, \lambda \) and \( \psi \), lead the class \( H^\alpha_\sigma(\gamma, \alpha, \lambda, \psi) \) to various following known subclasses.

1. \( H^\alpha_\sigma(\gamma, \alpha, \lambda, \psi) = H^\alpha_\sigma(\gamma, \alpha, \lambda, \psi) \) \( (\alpha \geq 0, \lambda \geq 0, \gamma \in C - \{0\}) \) (see Ramachandran, Prabhu and Magesh [19], inequality 2.11)
2. \( H^\alpha_\sigma(\gamma, 1 + 2\lambda, \psi, \psi) = H^\alpha_\sigma(\gamma, \lambda, \psi) \) \( (\lambda \geq 0, \gamma \in C - \{0\}) \) (see Tudor [8, Definition 2.1])
3. \( H^\alpha_\sigma(\gamma, 1 + 2\lambda, \lambda + \lambda^2 \phi^2 z^2, \lambda + \lambda^2 \phi^2 z^2) = H^\alpha_\sigma(\gamma, \lambda, \lambda) \) \( (\lambda \geq 0, \gamma \in C - \{0\}, -1 \leq B < A \leq 1) \) (see Bansal [12, Definition 1.1])
4. \( H^\alpha_\sigma(\gamma, 1 + 2\lambda, \lambda z^{3/2}, 1/2) \) for \( \gamma = e^{i\eta} \cos \theta \) \( 0 \leq \theta < \pi/2 \) is discussed and considered by Swaminathan [15, Definition 1.1])
5. \( H^\alpha_\sigma(\gamma, 1 + 2\lambda, \lambda z^{3/2}, 1/2) \) for \( \gamma = e^{i\eta} \cos \theta \) \( 0 \leq \pi < \pi < \pi/2 \) is discussed and considered by Lin [18]
6. \( H^\alpha_\sigma(\gamma, 1 + 2\lambda, \lambda z^{3/2}, 1/2) \) for \( \gamma = e^{i\eta} \cos \theta \) \( 0 \leq \pi < \pi/2 \) is discussed and considered by Lin [18]
7. \( H^\alpha_\sigma(\gamma, 1 + 2\lambda, \lambda z^{3/2}, 1/2) \) for \( \gamma = e^{i\eta} \cos \theta \) \( 0 \leq \pi < \pi/2 \) is discussed and considered by Lin [18]
8. \( H^\alpha_\sigma(\gamma, 1 + 2\lambda, \lambda z^{3/2}, 1/2) \) for \( \gamma = e^{i\eta} \cos \theta \) \( 0 \leq \pi < \pi/2 \) is discussed and considered by Lin [18]
9. \( H^\alpha_\sigma(\gamma, 1 + 2\lambda, \lambda z^{3/2}, 1/2) \) for \( \gamma = e^{i\eta} \cos \theta \) \( 0 \leq \pi < \pi/2 \) is discussed and considered by Lin [18]
10. \( H^\alpha_\sigma(\gamma, 1 + 2\lambda, \lambda z^{3/2}, 1/2) \) for \( \gamma = e^{i\eta} \cos \theta \) \( 0 \leq \pi < \pi/2 \) is discussed and considered by Lin [18]
11. \( H^\alpha_\sigma(\gamma, 1 + 2\lambda, \lambda z^{3/2}, 1/2) \) for \( \gamma = e^{i\eta} \cos \theta \) \( 0 \leq \pi < \pi/2 \) is discussed and considered by Lin [18]
12. \( H^\alpha_\sigma(\gamma, 1 + 2\lambda, \lambda z^{3/2}, 1/2) \) for \( \gamma = e^{i\eta} \cos \theta \) \( 0 \leq \pi < \pi/2 \) is discussed and considered by Lin [18]
13. \( H^\alpha_\sigma(\gamma, 1 + 2\lambda, \lambda z^{3/2}, 1/2) \) for \( \gamma = e^{i\eta} \cos \theta \) \( 0 \leq \pi < \pi/2 \) is discussed and considered by Lin [18]
14. \( H^\alpha_\sigma(\gamma, 1 + 2\lambda, \lambda z^{3/2}, 1/2) \) for \( \gamma = e^{i\eta} \cos \theta \) \( 0 \leq \pi < \pi/2 \) is discussed and considered by Lin [18]
15. \( H^\alpha_\sigma(\gamma, 1 + 2\lambda, \lambda z^{3/2}, 1/2) \) for \( \gamma = e^{i\eta} \cos \theta \) \( 0 \leq \pi < \pi/2 \) is discussed and considered by Lin [18]

**II. COEFFICIENT ESTIMATES FOR THE FUNCTION CLASS \( H^\alpha_\sigma(\gamma, \alpha, \lambda, \psi) \)**

In order to derive our results, we shall need the following lemma.

**Lemma 2:** [21] If \( h \in P \), then \( |h_k| \leq 2 \) for each \( k \), where \( P \) is the family of all functions \( h \), analytic in \( U \), for which

\[ \text{Re}(h(z)) > 0 \quad (z \in U), \]

where

\[ h(z) = 1 + c_1z + c_2z^2 + \ldots \quad (z \in U) \]

**Theorem 3:** If \( f \in H^\alpha_\sigma(\gamma, \alpha, \lambda, \psi) \), then

\[ |a_3| \leq \frac{\sqrt{|\gamma|^2 \lambda^2}}{\sqrt{1 + (\alpha + 2\lambda)(\mu + \lambda^2)}} \]

(8)
\[ |a_\gamma| \leq \frac{|\gamma|B_1}{\mu + 2\alpha + 2\lambda} + \frac{|\gamma|^2 B_1^2}{(\mu + a)^2} \]  

(9)

Proof: Since \( f \in H^2_\infty(\gamma, \alpha, \lambda, \psi) \), there exist two analytic functions \( u, v : U \to U \), with \( u(0) = 0 = v(0) \), satisfying

\[ 1 + \frac{1}{\gamma} \left( (1 - \alpha + 2\lambda) \left( \frac{f(z)}{z} \right)^\mu + (\alpha - 3\lambda) \frac{f'(z)}{f(z)} \right) + \lambda \left[ 1 + z \frac{f'(z)}{f(z)} \right] (f'(z))^{\mu - 1} < \varphi(u(z)) \]  

(10)

and

\[ 1 + \frac{1}{\gamma} \left( (1 - \alpha + 2\lambda) \left( \frac{g(w)}{w} \right)^\mu + (\alpha - 3\lambda) \frac{g'(w)}{g(w)} \right) + \lambda \left[ 1 + w \frac{g'(w)}{g(w)} \right] (g'(w))^{\mu - 1} < \varphi(v(w)). \]  

(11)

Define the function \( c \) and \( d \) by

\[ c(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \]  

(12)

and

\[ d(z) = \frac{1+v(z)}{1-v(z)} = 1 + d_1 z + d_2 z^2 + d_3 z^3 + \cdots. \]  

(13)

or equivalently

\[ u(z) = \frac{c(z)+1}{c(z)-1} = \frac{1}{2} \left( c_1 z + \left( c_2 + \frac{c_1^2}{2} \right) z^2 + \left( c_3 + \frac{c_1 c_2}{2} \right) z^3 + \cdots \right) \]  

(14)

and

\[ v(z) = \frac{d(z)+1}{d(z)-1} = \frac{1}{2} \left( d_1 z + \left( d_2 + \frac{d_1^2}{2} \right) z^2 + \left( d_3 + \frac{d_1 d_2}{2} \right) z^3 + \cdots \right). \]  

(15)

it is clear that \( c(z) \) and \( d(z) \) are analytic having positive real part in \( U \) and \( c(0) = 1 = d(0) \). \( |c| \leq 2 \) and \( |d| \leq 2 \). In view of (10), (11), (14) and (15), clearly

\[ 1 + \frac{1}{\gamma} \left( (1 - \alpha + 2\lambda) \left( \frac{f(z)}{z} \right)^\mu + (\alpha - 3\lambda) \frac{f'(z)}{f(z)} \right) + \lambda \left[ 1 + z \frac{f'(z)}{f(z)} \right] (f'(z))^{\mu - 1} < \varphi \left( \frac{c(z)-1}{c(z)+1} \right) \]  

(16)

and

\[ 1 + \frac{1}{\gamma} \left( (1 - \alpha + 2\lambda) \left( \frac{g(w)}{w} \right)^\mu + (\alpha - 3\lambda) \frac{g'(w)}{g(w)} \right) + \lambda \left[ 1 + w \frac{g'(w)}{g(w)} \right] (g'(w))^{\mu - 1} < \varphi \left( \frac{d(w)-1}{d(w)+1} \right). \]  

(17)

Using (14) and (15) together with (5), we get

\[ \varphi \left( \frac{c(z)-1}{c(z)+1} \right) = 1 + \frac{1}{\gamma} B_1 c_1 z + \left( \frac{1}{\gamma} B_1 \left( c_2 - \frac{1}{2} c_1^2 \right) + \frac{1}{4} B_2 c_1^2 \right) z^2 + \cdots \cdots \cdots \]  

(18)

and

\[ \varphi \left( \frac{d(w)-1}{d(w)+1} \right) = 1 + \frac{1}{\gamma} B_1 d_1 z + \left( \frac{1}{\gamma} B_1 \left( d_2 - \frac{1}{2} d_1^2 \right) + \frac{1}{4} B_2 d_1^2 \right) z^2 + \cdots \cdots \cdots \]  

(19)

It follows from (14), (15), (18) and (19) that

\[ \frac{1}{\gamma} (\mu + \alpha) a_\gamma = \frac{1}{\gamma} B_1 c_1 \]  

(20)

\[ \frac{(2x+2\lambda+\mu)}{\gamma} a_\gamma + \frac{(\mu-1)(2x+2\lambda+\mu)}{2\gamma} a_\gamma^2 = \frac{1}{\gamma} B_1 \left( c_2 - \frac{1}{2} c_1^2 \right) + \frac{1}{4} B_2 c_1^2 \]  

(21)

\[ \frac{1}{\gamma} (\mu + \alpha) a_\gamma = \frac{1}{\gamma} B_1 d_1 \]  

(22)

\[ \frac{(2x+2\lambda+\mu)}{\gamma} a_\gamma + \frac{(\mu-1)(2x+2\lambda+\mu)}{2\gamma} a_\gamma^2 = \frac{1}{\gamma} B_1 \left( d_2 - \frac{1}{2} d_1^2 \right) + \frac{1}{4} B_2 d_1^2 \]  

(23)

From (20) and (22), we have

\[ c_1 = -d_1 \]  

(24)

and

\[ \frac{\theta(\mu+\alpha)^2}{\gamma} a_\gamma^2 = B_1^2 \left( c_1^2 + d_1^2 \right). \]  

(25)

from (21), (23) and (25), we get

\[ a_\gamma^2 \leq \frac{\gamma^2 \theta(\mu+\alpha)^2}{2\gamma} \frac{B_1^2 (c_1^2 + d_1^2)}{\theta(\mu+\alpha)^2}. \]  

(26)

by using Lemma, we get the desired estimate on \( |a_\gamma| \) as

\[ |a_\gamma| \leq \frac{|\gamma|B_1}{\mu + 2\alpha + 2\lambda} + \frac{|\gamma|^2 B_1^2}{(\mu + a)^2} \]  

by subtracting (21) and (23) yields
by using Lemma, we get the desired estimate on $|a_3|$ as

$$|a_3| \leq \frac{|y|}{\mu + 2\alpha + 2\lambda} + \frac{|y|^2}{(\mu + \alpha)^2}.$$

Remark 4:- Taking $\mu = 1$ in Theorem 3, we obtain the corresponding result given earlier by Ramachandran [19]. For $\mu = 1$ and $\lambda = 0$ in Theorem 3 we have result of Tudor [8]. For $\mu = 1$, $\gamma = 1$ and $\alpha = 1 + 2\lambda$ in Theorem 3 we have result of Kumar and Ravichandran [9]. For $\mu = 1$, $\gamma = 1$, $\lambda = 0$ and $\alpha = 1$ in Theorem 2.2, we have result by Ali, Lee, Ravichandran and Supramaniam [11].

If we set $\varphi(z) = \frac{1 + \alpha z}{1 + \beta z} - 1 \leq B < A \leq 1$ in the class $H_\mu^\gamma(y, \alpha, \lambda, \psi)$ we have $H_\mu^\gamma(y, \alpha, \lambda, \frac{1 + \alpha z}{1 + \beta z})$ and defined as

$$1 + \frac{1}{\gamma} \left(1 - \alpha + 2\lambda \right) \left(\frac{f(z)}{z}\right)^\mu + (\alpha - 3\lambda) \left(\frac{f'(z)}{f(z)}\right)^\mu + \lambda \left[1 + z \left(\frac{f''(z)}{f'(z)}\right) \left(f'(z)\right)^\mu - 1\right] < \frac{1 + \alpha z}{1 + \beta z},$$

and

$$1 + \frac{1}{\gamma} \left(1 - \alpha + 2\lambda \right) \left(\frac{g(w)}{w}\right)^\mu + (\alpha - 3\lambda) \left(\frac{g'(w)}{g(w)}\right)^\mu + \lambda \left[1 + w \left(\frac{g''(w)}{g'(w)}\right) \left(g'(w)\right)^\mu - 1\right] < \frac{1 + \alpha z}{1 + \beta z},$$

where $g(w) = f'(w)$

Corollary 5:- If $f \in H_\mu^\gamma(y, \alpha, \lambda, \frac{1 + \alpha z}{1 + \beta z})$, then

$$|a_2| \leq \frac{|\gamma|}{\sqrt{\gamma (\mu + 1)(\mu + 2\alpha + 2\lambda)}}$$

and

$$|a_3| \leq \frac{|\gamma|^2}{\mu + 2\alpha + 2\lambda}.$$


