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On Some Growth Curve Modeling: Approximation Theory and Applications

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Abstract: Mathematical models of growth have been developed a long period of time. Estimating the lag time in the growth process is a practically important problem. Any sigmoidal function can be good illustration for the concept of lag time. The Stannard growth model is described by 4 free parameters, each contributing to the characteristics of the curve: an initial lag or period of slow growth; a period of rapid exponential growth; a period of reduced growth rate. In this note we provide more precise estimates for the one–sided Hausdorff approximation of the Heaviside step–function by sigmoidal Stannard function - ($t_{new-lag}$). Numerical examples, illustrating our results are given, too.

Keywords— *sigmoidal Stannard function; interval step function; Hausdorff distance; upper and lower bounds; lag time.*

I. INTRODUCTION

Several sigmoidal functions (logistic [32],[31], Gompertz [8], Richards [24], [30], [34], [15], Chapman–Richards (based on the Von Bertalanffy's approach [4]), Schnute [25], and Stannard [28], [12], [35], [23]) were compared to describe a growth curve. Growth curves are found in a wide range of disciplines, such as biology, chemistry and medical science. Estimating the lag time in the growth process is a practically important problem. Nevertheless, any sigmoidal function can be good illustration for the concept of lag time. The growth model is described by free parameters, each contributing to the characteristics of the sigmoidal function. These parameters may be useful for describing biologically relevant metrics as a lag phase, the growth phase, and the plateau phase. The lag time - t_{lag} (see Fig. 1) is estimated by extending the tangent at inflection point to the initial baseline. The Stannard curve is described by 4 free parameters, each contributing to the characteristics of the curve: an initial lag or period of slow growth; a period of rapid exponential growth; a period of reduced growth rate. The Stannard function finds applications in many scientific fields, including population dynamics, bacterial growth, population ecology, plant biology, chemistry, demography, financial mathematics, statistics and fuzzy set theory. For some modelling aspects and parameter estimations, see [3], [27], [2], [29], [33], [6], [13]. The alternative definition of the t'_{lag} is given by Arosio, Knowles and Linse in [2]. The t'_{lag} is defined as the point in time where the signal relative to the pre–transition baseline has reached 10% of the amplitude of the transition. In this note we prove more precise estimates for the one–sided Hausdorff approximation of the interval Heaviside step–function by sigmoidal Stannard function - ($t_{new-lag}$). Let us point out that Hausdorff distance is the most natural measuring criteria for the approximation of bounded discontinuous function [1], [16].

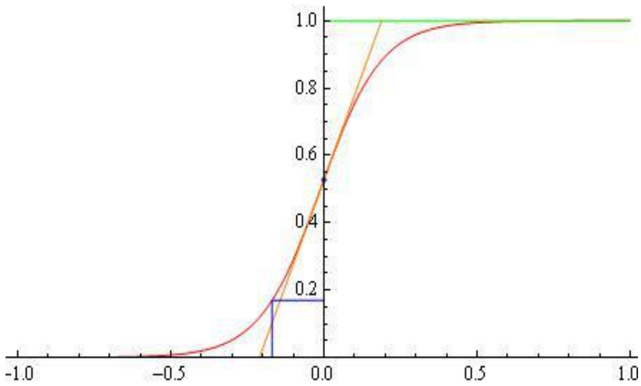

Figure 1: Definitions: a) t_{lag} - is estimated by extending the tangent at inflection point to the initial baseline; b) $t_{new-lag}$ - the one–sided Hausdorff approximation of the Heaviside step–function by sigmoidal Stannard function.

II. PRELIMINARIES

Definition 1. Define the interval Heaviside step function as:

$$h_0(t) = \begin{cases} 0, & \text{if} \quad t < 0, \\ [0, M] & \text{if} \quad t = 0, \\ M, & \text{if} \quad t > 0. \end{cases} \qquad (1)$$

Definition 2. Define the sigmoidal Stannard function $S^*(t;\beta,k,m,M)$ on R as:

$$S^*(t;\beta,k,m,M) = \frac{M}{\left(1+e^{\frac{-(\beta+kt)}{m}}\right)^m}, \qquad (2)$$

where M is the upper asymptote; β is the growth displacement; k is the growth rate; m is the slope of growth.

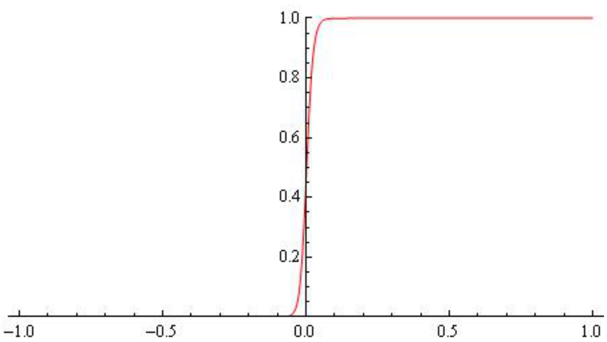

Figure 2: Approximation of the Heaviside step function by Stannard sigmoidal function about Hausdorff distance: $k = 100$, $m = 1.2$, $\beta = 0.1$, $M = 1$; Hausdorff distance $d = 0.034057$.

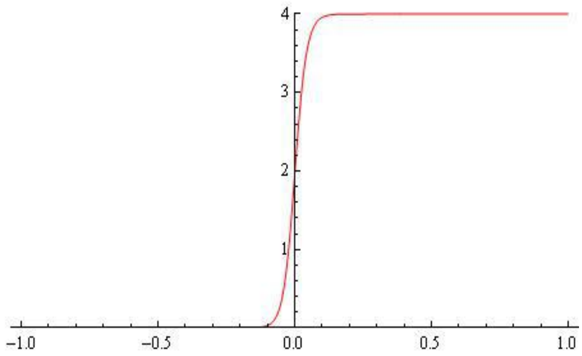

Figure 3: Approximation of the Heaviside step function by Stannard sigmoidal function about Hausdorff distance: $k=50$, $m=1.1$, $\beta=0.1$, $M=4$; Hausdorff distance $d=0.0796847$.

Definition 3. The Hausdorff distance (H-distance) $\rho(f,g)$ between two interval functions f,g on $\Omega\subseteq\mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega\times\mathbb{R}$ [9], [26]. More precisely,

$$\rho(f,g)=\max\{\sup_{A\in F(f)}\inf_{B\in F(g)}\|A-B\|, \quad (3)$$

$$\sup_{B\in F(g)}\inf_{A\in F(f)}\|A-B\|\},$$

wherein $\|.\|$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $\|(t,x)\|=\max\{|t|,|x|\}$; hence the distance between the points $A=(t_A,x_A)$, $B=(t_B,x_B)$ in $\mathbb{R}^2$ is $\|A-B\|=max(|t_A-t_B|,|x_A-x_B|)$.

III. MAIN RESULTS

We study the Hausdorff approximation of the Heaviside step function $h_0(t)$ by sigmoidal Stannard function $S^*(t;\beta,k,m,M)$ and find an expression for the error of the best one–sided approximation. The Hausdorff distance d satisfies the relation (see, Fig. 1)

$$S^*(-d;\beta,k,m,M)=\frac{M}{\left(1+e^{\frac{-(\beta-kd)}{m}}\right)^m}=d. \quad (4)$$

The following Theorem gives upper and lower bounds for d

Theorem 3.1. For the one–sided Hausdorff distance $d=d(k,\beta,m,M)$ between the function $h_0(t)$ and the Stannard function $S^*(t;\beta,k,m,M)$ the following inequalities hold for $kM\geq 3e^{\frac{\beta}{m}}\left(1+e^{-\frac{\beta}{m}}\right)^{m+1}$:

$$d_l=\frac{M}{\left(1+e^{-\frac{\beta}{m}}\right)^m}\frac{1}{2\left(1+\frac{kM}{e^{\frac{\beta}{m}}\left(1+e^{-\frac{\beta}{m}}\right)^{m+1}}\right)}<d<$$

$$<\frac{M}{\left(1+e^{-\frac{\beta}{m}}\right)^m}\frac{\ln\left(2\left(1+\frac{kM}{e^{\frac{\beta}{m}}\left(1+e^{-\frac{\beta}{m}}\right)^{m+1}}\right)\right)}{2\left(1+\frac{kM}{e^{\frac{\beta}{m}}\left(1+e^{-\frac{\beta}{m}}\right)^{m+1}}\right)}=d_r. \quad (5)$$

Proof. We need to express d in terms of k, β, m and M, using (4). Let us examine the function

$$F(d)=\frac{M}{\left(1+e^{\frac{-(\beta-kd)}{m}}\right)^m}-d.$$

From $F'(d)<0$ we conclude that the function F is strictly monotonically decreasing. Consider function

$$G(d)=\frac{M}{\left(1+e^{-\frac{\beta}{m}}\right)^m}-\left(1+\frac{kM}{e^{\frac{\beta}{m}}\left(1+e^{-\frac{\beta}{m}}\right)^{m+1}}\right)d.$$

From Taylor expansion

$$-d+\frac{M}{\left(1+e^{\frac{-(\beta-kd)}{m}}\right)^m}=$$

$$\frac{M}{\left(1+e^{-\frac{\beta}{m}}\right)^m}-\left(1+\frac{kM}{e^{\frac{\beta}{m}}\left(1+e^{-\frac{\beta}{m}}\right)^{m+1}}\right)d+O(d^2)$$

we obtain $G(d)-F(d)=O(d^2)$. Hence $G(d)$ approximates $F(d)$ with $d\to 0$ as $O(d^2)$ (see, Fig. 4). In addition $G'(d)<0$. Further, for $kM\geq 3e^{\frac{\beta}{m}}\left(1+e^{-\frac{\beta}{m}}\right)^{m+1}$ we have

$$G(d_l) = \frac{M}{\left(1+e^{-\frac{\beta}{m}}\right)^m} - \frac{1}{2}\frac{M}{\left(1+e^{-\frac{\beta}{m}}\right)^m} > 0,$$

$$G(d_r) = \frac{M}{\left(1+e^{-\frac{\beta}{m}}\right)^m} \times$$

$$\left(1-\frac{1}{2}\ln\left(2\left(1+\frac{kM}{e^{\frac{\beta}{m}}\left(1+e^{-\frac{\beta}{m}}\right)^{m+1}}\right)\right)\right) < 0.$$

This completes the proof of the theorem.

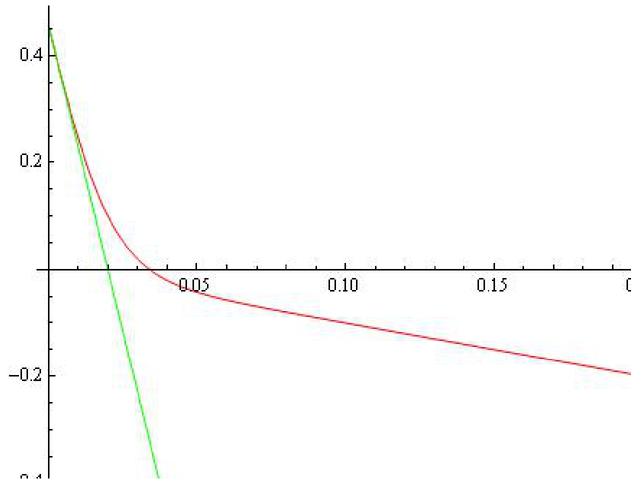

Figure 4: The functions $F(d)$ and $G(d)$ for $k=100$, $\beta=0.1$, $m=1.2$, $M=1$.

The "new" lag time is then given in terms of the one–sided Hausdorff distance - d.

IV. COMPUTATIONAL ISSUES

Some computational examples using relations (5) are presented in Table 1. The last column of Table 1 contains the values of d computed by solving the nonlinear equation (4).

β	k	m	M	d_l	d_r	d
0.0005	100	0.99	4	0.0099417	0.0526419	0.0448138
0.1	50	1.1	4	0.020513	0.0935269	0.0796847
0.2	100	2	2	0.010138	0.0405115	0.0385117
0.1	100	1.2	1	0.00997894	0.0381641	0.034057
0.15	25	1.1	1	0.0366494	0.0958873	0.0947611
0.2	500	0.99	1	0.00220615	0.0121895	0.00966098
0.1	5000	1	1	0.000210348	0.00164542	0.00134239

Table 1: Bounds for d computed by equation (4) for various β, k, m and M

```
k = Input[" k"]; (*50 *)
Print[" k = ", k];
m = Input[" m"]; (*1.1 *)
Print[" m = ", m];
β = Input[" β"]; (*0.1 *)
Print[" β = ", β];
M = Input[" M"]; (*4 *)
Print[" M = ", M];
Print["The following nonlinear equation is used to determination of
the one-sided Hausdorff distance between Heaviside function and
Stannard growth curve - d (the new_lag_time): "];
Print["M/(1+Exp[-(β-k*d)/m])^m-d=0"];
FindRoot[M / (1 + Exp[-(β - k * d) / m]) ^ m - d, {d, 0}]

 k = 50
 m = 1.1
 β = 0.1
 M = 4
The following nonlinear equation is used to determination of
the one-sided Hausdorff distance between Heaviside function and
Stannard growth curve - d (the new_lag_time):
M/(1+Exp[-(β-k*d)/m])^m-d=0
{d → 0.0796847}
```

Figure 5: Simple module implemented in programming environment CAS Mathematica for calculation of the value of the one–sided Hausdorff distance d between the Heaviside step function and the sigmoidal Stannard function.

Simple module in CAS Mathematica for calculation of the value of the one–sided Hausdorff distance d between the Heaviside step function and the sigmoidal Stannard function is visualized on Figure 5.

Some computational examples are presented on Figure 6.

```
Manipulate[Dynamic@Show[Plot[f[t], {t, -1, 1}, LabelStyle → Directive[Green, Bold],
   PlotLabel → M / (1 + Exp[-(β + k * t) / m]) ^ m],
  PlotRange → {Automatic, {0, M}}], {{M, 0.01}, 0.01, 10, Appearance → "Open"},
 {{β, 0.0005}, 0.0005, 10, Appearance → "Open"}, {{k, 0.1}, 0.1, 200, Appearance → "Open"},
 {{m, 0.0001}, 0.0001, 10, Appearance → "Open"},
 Initialization :→ (f[t_] := M / (1 + Exp[-(β + k * t) / m]) ^ m)]
```

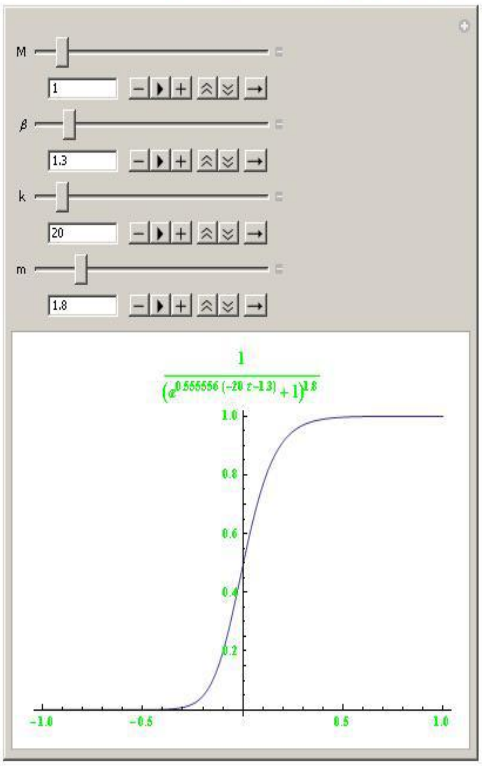

Figure 6: An example of the usage of dynamical and graphical representation. The plots are prepared using CAS Mathematica.

Definition 4. Define the 6–parameters Stannard growth function $S_1^*(t)$ on $\mathbb{R}$ as:

$$S_1^*(t;a_1,b_1,a_2,\beta,k,m)=a_1+b_1t+\frac{a_2}{\left(1+e^{\frac{-(\beta+kt)}{m}}\right)^m},\qquad(6)$$

where the linear part a_1+b_1t described the lag–phase (see Fig. 7).

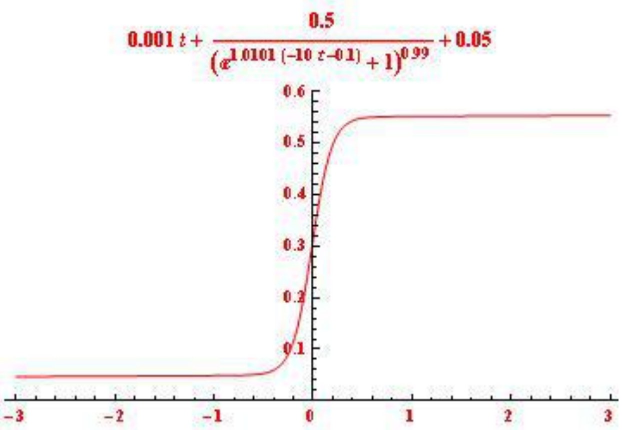

Figure 7: 6–parameters Stannard growth function (6) with: $k=10$, $m=0.99$, $\beta=0.1$, $a_1=0.05$, $b_1=0.001$, $a_2=0.5$.

Definition 5. Define the 7–parameters Stannard growth function $S_2^*(t))$ on $\mathbb{R}$ as:

$$S_2^*(t;a_1,b_1,a_2,\beta,k,m)=a_1+b_1t+\frac{a_2+b_2t}{\left(1+e^{\frac{-(\beta+kt)}{m}}\right)^m},\qquad(7)$$

where the linear part a_2+b_2t described the equilibrium baselines (see Fig. 8).

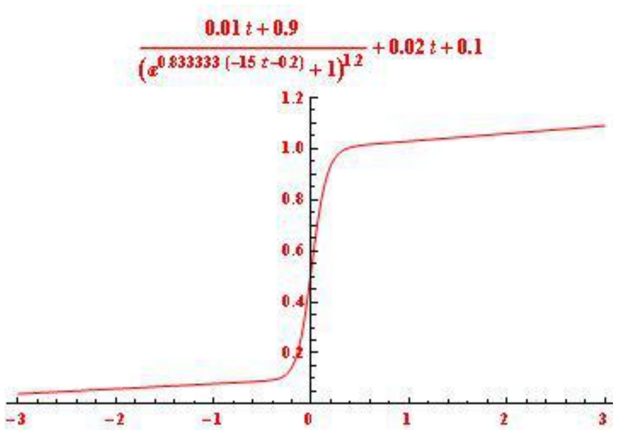

Figure 8: 7–parameters Stannard growth function (7) with: $k=15$, $m=1.2$, $\beta=0.2$, $a_1=0.1$, $b_1=0.02$, $a_2=0.9$, $b_2=0.01$.

Definition 6. Define the shifted interval Heaviside step function h_{t_m} as:

$$h_{t_m}(t)=\begin{cases}0, & \text{if } t<t_m,\\ [0,M], & \text{if } t=t_m,\\ M, & \text{if } t>t_m.\end{cases}\qquad(8)$$

Definition 7. Define the shifted Stannard growth function $S_{t_m}^*(t)$ with jump at point t_m as:

$$S_{t_m}^*(t;\beta,k,m,M)=\frac{M}{\left(1+e^{\frac{-(\beta+k(t-t_m))}{m}}\right)^m}.\qquad(9)$$

We next focus on the approximation of the shifted interval Heaviside step function h_{t_m} by shifted Stannard growth function $S_{t_m}^*(t)$.

V. FITTING THE NONLINEAR SHIFTED STANNARD GROWTH MODEL AGAINST EXPERIMENTAL OIL PALM DATA [12], [7]

Example. The oil palm yield growth data is given in Table 2.

Year	*Weight*	*The appropriate fitting by shifted Stannard function* (9)
4	11.78	11.3382
5	18.43	18.1988
6	25.21	24.7753
7	30.78	29.9221
8	33.03	33.4182
9	35.66	35.5881
10	36.96	36.8639
11	37.97	37.591
12	38.04	37.9981
13	39.20	38.2239
14	36.50	38.3484
15	37.21	38.4108
16	39.97	38.4544
17	38.45	38.475

Table 2: The oil palm yield data [12], [7]

The appropriate fitting of the experimental data by the shifted Stannard growth function $S_{t_m}^*(t)$ (9) with $M=38.5$,

$\beta = 2.11$, $k = 1.36$, $m = 2.26$, $t_m = 5$ is visualized on Figure 9 (see also the last column of Table 2).

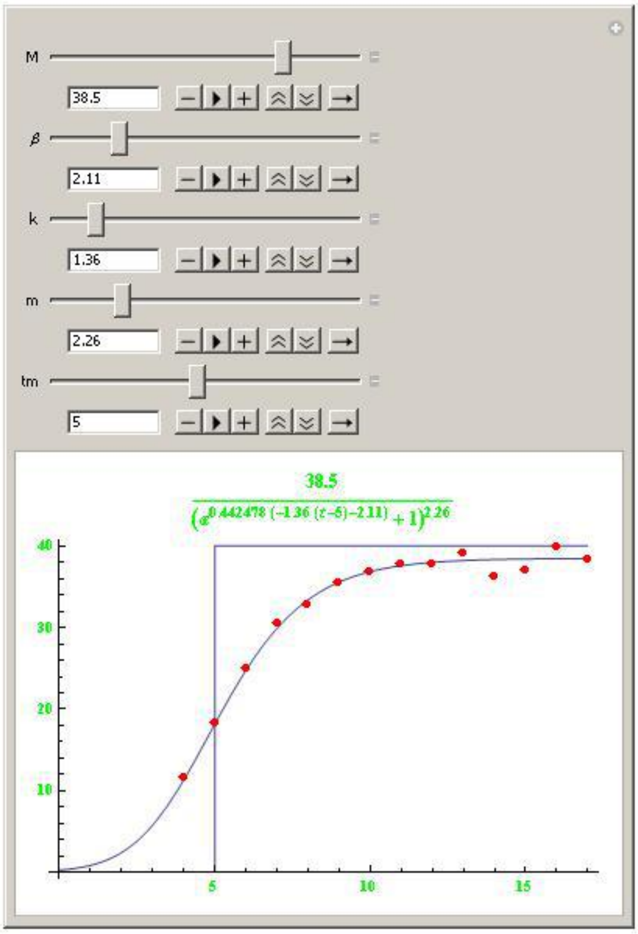

Figure 9: The appropriate fitting of experimental data by the shifted Stannard growth function $S^*_{t_m}(t)$ with $M = 38.5$, $\beta = 2.11$, $k = 1.36$, $m = 2.26$, $t_m = 5$.

CONCLUSION REMARKS

New estimates for the H–distance between a interval Heviside step function and its best approximating Stannard function are obtained. We propose a modified new–lag–time in terms of Hausdorff distance - d .

Based on the methodology proposed in the present note, the reader may formulate the corresponding approximation problems on his/her own.

On a number of computational examples we demonstrate the applicability of the Stannard growth function to approximate the Heaviside step function and consequently to be employed in fitting time course experimental data related to population dynamics.

The Hausdorff approximation of the interval step function by the logistic and other sigmoid functions is discussed from various approximation, computational and modelling aspects in [5], [14], [21], [18], [10], [11], [19], [20], [17], [22].

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