International Journal of Trend in Research and Development, Volume 10(6), ISSN: 2394-9333
www.ijtrd.com

The Evolution of Axiomatic Methods and Their Impact on Mathematics Education

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Abstract: From Euclid's "Elements of Geometry" to Hilbert's "Fundamentals of Geometry", and then to the study of mathematical foundations, people have put forward higher requirements for the rigor of mathematics.The evolution and improvement of axiomatic methods are concrete manifestations of mathematical rigor, which is a requirement for the development of mathematics itself.Excessive pursuit of mathematical rigor has had a significant impact on mathematical education：The relationship between mathematics and practice is artificially separated, the role of the history of mathematics is neglected while the logical factor is emphasized, the result content is emphasized, and the process knowledge is neglected.

Keywords: *Axiomatic Method; Rigor; History Of Mathematics; Logic*

1. The Historical Evolution of Axiomatic Method

Before the 7th century BC, the so-called Greek geometry was only limited to the solution of specific problems, and its solution process was very rough and based solely on experience, and there was no rigor at all.After the 7th century BC, Greek geometers began to devote themselves to the study of geometry.From the Ionian School represented by Thales to the Pythagorean School, from the Platonic School to the Ordovician School, through the continuous efforts and exploration of generations of mathematicians, the study of Greek geometry has reached its peak.This not only accumulated a wealth of geometric knowledge for later scholars, but also began to notice some laws that deductive reasoning should follow.After Aristotle's refinement and generalization, syllogism was established, laying the foundation for formal logic and making it possible to systematize the large amount of geometric materials accumulated by predecessors.

1.1 Shortcomings and defects of euclideangeometry

In the 3rd century BC, the outstanding Greek geometer Euclid organized the geometric knowledge obtained by his predecessors based on their logical connections. On the other hand, he modified the original lax proofs and completed the great work " Elements of Geometry" on the basis of systematically summarizing the knowledge of his predecessors.The "Elements of Geometry" not only collects the great achievements of predecessors, but also states the content of this discipline with a quite strict logical deduction system, becoming a shining example of mathematical rigor.Over the past 2000 years, "Elements of Geometry" has faced all challenges and still exists in the world, and its strict proof is considered perfect and flawless.In fact, according to the strict standard of modern mathematics, Euclid's Elements of Geometry is not strict enough, which is mainly reflected in the following aspects:(1) Lack of rigor in terms of definitions; (2) The incompleteness of axioms, that is, the lack of proper axioms; (3) There are loopholes in logic.This is related to (2), as there is a lack of proper axioms. When proving certain propositions, there is a lack of necessary basis, relying heavily on visual representation to make judgments, which clearly conflicts with strict logical proof.Due to the shortcomings mentioned above, with the further development of mathematics, the errors and defects in the "Elements of Geometry"will inevitably become increasingly exposed, forcing people to re-examine, modify, and improve it, so as to achieve a higher level of rigor in the axiomatic system.

1.2 Hilbert's contribution to axiomatic methods

In the 19th century, the field of mathematical research rapidly expanded at an unprecedented pace, especially with the emergence of non Euclidean geometry due to the study of the fifth postulate in the "Elements ofGeometry ". This not only increased people's confidence in axiomatic methods, but also laid the foundation for further rigor of axiomatic systems.Many mathematicians have devoted themselves to the study of axiomatic systems. Cantor and Dedekin independently established the continuous axiom, and later Pascal formulated the sequential axiom. Finally, Hilbert completed the significant work "Geometry Foundations" on the basis of previous research results, establishing a very strict formal foundation for Euclidean axiomatic systems.Specifically,(1) The basic concepts include three types of objects (the first type is points, the second type is lines, and the third type is planes) and the basic relationships between them. This is no longer the concrete objects and relationships that Euclidean geometry refers to, but rather a more general and advanced abstraction of various geometric objects and relationships.Therefore, the axiomatic system established by Hilbert fundamentally breaks free from the constraints of object intuitiveness.(2)They are supplemented and divided into five groups of twenty axioms according to the meanings of the basic relations between the objects, in other words, the five groups of axioms define the strict meanings of the basic relations between the objects.Thus, the axiom system of Euclidean geometry was improved, abandoning the intuitive or experiential arguments in Euclidean geometry, and ultimately transforming the "object axiom deduction" system of Euclidean geometry into a "hypothesis deduction" system.(3)While Hilbert was conducting fundamental research on geometry, the rigor of axiomatic systems was no longer limited to the accuracy of individual concepts and the rigor of individual proposition proofs, but also involved the overall rigor of axiomatic systems.Especially after the emergence of Russell's paradox, it sparked research on mathematical foundations. In order to break free from the dilemma of mathematical foundation research, Hilbert proposed a specific plan for mathematical foundation research.The overall rigor standard for axiomatic systems was proposed along with the implementation of Hilbert's specific planning. In fact, the formal system transformed from classical mathematical theory is an axiomatic system. Hilbert further pointed out that as long as the axiomatic system in the formal system is guaranteed to be free of contradictions, independence, and completeness, the reliability and rigor of the entire formal system can be guaranteed.

Although this absolute strictness standard ultimately did not materialize, Hilbert's idea of organizing classical mathematical theories into a formalized system greatly promoted the development of axiomatic methods, pushing them to a new stage.In the formal method and proof theory, the axiomatic method is further symbolized and purely formalized, so that the logical structure of mathematical theory is fully exposed, which lays the foundation for in-depth research on the internal relations of various mathematical theories.Due to this formalized axiom system completely breaking free from the constraints of specific objects, it brings a certain degree of freedom to mathematical research, that is, people can freely establish various other formalized axiom systems, greatly expanding the scope of mathematical research.

2. The Influence of Axiomatic Methods on Mathematics and Mathematics Education

The research on axiomatic systems has the longest history, the most effort, and the most typical achievements. The axiomatic methods established as a result are particularly favored by people, not only because they provide a typical strict standard for people, but also because they often crystallize the final product of mathematical ideas. Influenced by this method, most mathematical textbooks nowadays unilaterally pursue the rigor of this method in form. The corresponding knowledge systems in the textbooks present a deductive knowledge structure, and knowledge systems with this organizational form only emphasize the logical connections between knowledge. Students only see the starting point of logical construction - basic concepts or principles, and the end point of logical reasoning - inference. As American mathematicians Davis and Hersh pointed out in their book "Mathematical Experience":“Nowadays, a considerable number of mathematics textbooks have a neurotic and suffocating characteristic, in which they systematically and tenaciously pursue a fixed goal. Once this goal is achieved,people don't feel excited, they feel like a tiger's head and a snake's tail. This type of book does not have much to say about why or how important this purpose is, but rather may talk about how this purpose can now become a starting point for achieving other deeper purposes, ……。If you want to blame, blame Euclid, because this tendency already exists in his place.”[1] In fact, textbooks with this structure often give people the impression that mathematics seems to be a "self developing closed system" that does not require external motivation. On the basis of previous achievements, relying solely on rigorous logical reasoning can deduce all mathematical theories. The rigor reflected by the axiomatic method has had the following impact on mathematical education.

2.1 One-sided emphasis on the self-development of mathematics, neglecting the relationship between mathematics and practice

As we all know, mathematics originates from practice. From the generation of the earliest concept of number and the simplest geometric figure to the establishment of the extensive and profound modern mathematical theory system, it has a certain connection with the real world or human social practice directly or indirectly. In terms of mathematical objects, there is a considerable portion that has a clear intuitive background, which is directly extracted from the real world through abstract thinking, while the other portion is constructed due to the logical development of mathematics itself. Even these objects are often more closely integrated with reality at a higher level.“Recognizing the relationship between mathematics and practice, one should grasp the two directions of mathematical creation, be adept at abstracting new concepts and theories from the process of solving practical problems, and seek new methods. On the basis of existing concepts, theories, and methods, we must constantly invent and create more abstract and general mathematical concepts and related theories, thereby promoting the inheritance and further development of mathematical culture.”[2]Therefore, ignoring the relationship between mathematics and practice will not enable students to correctly deal with the relationship between mathematics and the physical world and the spiritual world. There will be a formalism understanding of mathematical objects, and students will not recognize the extensive application of mathematics, which will lead to the formation of a wrong view of "mathematics is useless", and seriously inhibit the cultivation of students' ability to integrate theory with practice.

2.2 One-sided emphasis on logical factors and neglect of historical factors

“Logic is a reflection of history, and objective history is the foundation of logic. History reflects mathematicians' thought journey and cognitive process of mathematical creation, which is often influenced by the political, philosophical, religious and other concepts at that time, with twists and turns and contingency; In mathematics, the essence and laws of historical processes can be revealed by the knowledge system that reflects the deductive structure through the use of logical reasoning rules, but they may not necessarily be fully and truly reflected.”[3]Secondly, logic and history are consistent and generally correct, but in terms of specific concepts or propositions, the two sometimes exhibit inconsistency.For example, in textbooks, generally, exponential function is defined first, and then logarithmicfunction is defined with the help of the concept of inverse function. From the logical construction order of the two, exponential function comes first and logarithmic function comes last.But in history, because of the practical needs of the development of astronomy, people first invented the logarithmic function, and then exponential function.For another example, it is easy to see from the current Advanced Mathematics textbook that limit theory is the basis of calculus. It seems that there is limit theory first, followed by calculus, while the actual situation in history is just the opposite.Third, there are many important concepts in mathematics, such as irrational numbers, imaginary numbers, non Euclidean geometry, axiomatic set theory, etc. From their generation to their acceptance, they have gone through a long and tortuous process. This process is actually the process of the change and development of people's mathematical concepts. Reproducing this historical process will undoubtedly eliminate students' "confusion" in understanding these concepts or theories, So as to have a comprehensive understanding and deeper understanding of these mathematical knowledge.Ignoring historical factors often leads to a negative tendency of "seeing only trees but not forests", which negates the important role of history in mathematics education and even makes erroneous judgments that do not conform to historical facts.

2.3 Only focusing on results oriented content, neglecting procedural knowledge

In terms of its final result, mathematical theory reflects the deductive nature of mathematics, while in terms of its formation process, it reflects the other side of mathematics, namely the inductive nature of mathematics. Mathematician Maurice Klein once described: “Mathematicians must spend decades or even centuries creating materials in order to ensure a deductive knowledge system like Euclidean geometry. And unlike logical organizations, this creative work is not a gradual progression from one proof to another, where each proof is supported by axioms and previously established conclusions. This creative process involves exploration, negligence, speculation andassumptions.Imagination,intuition,prediction,insight,experimentation, opportunity, luck, hard work, and great patience are all used to master a key concept, form a guess, and find a proof. Overall, mathematical creativity lies in 'using one's own wisdom to do the things one is most bored of, while grasping all possibilities'. No logically correct guidance can tell the mind how to think. The fact that great mathematicians set out to solve a problem and failed, while another came to solve it and succeeded, indicates how many people's mental labor was incorporated into creation, which is not the systematic and orderly reasoning demonstrated in the final proof.”[4] “The satisfaction that a mathematician can achieve in creative work, the excitement of hunting, the vibrato of discovery, the awareness of achievement, and the pride of success are all much more and more intense than what he can achieve in the final organization of proof based on deductive patterns.”[5] In the traditional teaching process, overemphasizing the process of mathematization and only focusing on the results of mathematical knowledge will overlook the process of knowledge reconstruction, block the source and flow of mathematical knowledge, and inevitably form a dogmatic teaching method that is monotonous, erasing students' imagination and creativity, and failing to achieve the expected teaching effect.

References

[1] P. Translated by J. Davis, R. Hesh, Qian Wang, Xiaoqun Yu, etc. Mathematical Experience [M]. Nanjing: Jiangsu Education Press, 1991,76.

[2] Xilian Fu, Yufeng Zhang. Dialectical Thought of calculus and Its Cultural Value Perspective. Coal Higher Education, 2019, 37 (5): 101.

[3] Dingqiu Wu, Zheng Chen. On the Consistency and Inconsistency between Logic and History [J]. Journal of Renmin University of China, 1995,9 (2): 34.

[4] Kapur.J.N. Trans. QingrenWang.Mathematicians talk about the essence of mathematics, Beijing: Peking University Press, 1987, 320-321.