Projective change between two Special (α,β)- Finsler Metrics

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Abstract: In Finsler space we see special metrics such as Randers metric, Kropina metric and Matsumoto metric., etc. Projective change between two Finsler metrics arise from Information Geometry. Such metrics have special geometric properties and will play an important role in Finsler geometry. In this paper,we are going to study class of Projective change between two $(\alpha,\beta)-$ metrics, which are defined as the sum of a Riemannian metric and $1-form$.

Keywords: Finsler metric, Special Finsler metric,$(\alpha,\beta)-$metric, Douglas Space, Geodesic, Spray coefficients, Projectively related metric, Projective change between two metrics.

I. INTRODUCTION

- The projective change between two Finsler spaces have been studied by many authors ([1], [5], [6], [8]).
- An interesting result concerned with the theory of projective change was given by Rapscak's paper. He proved necessary and sufficient conditions for projective change.
- S. Bacso and M. Matsumoto [2] discussed the projective change between Finsler spaces with $(\alpha,\beta)-metric$.
- H. S. Park and Y. Lee [6] studied on projective changes between a Finsler space with $(\alpha,\beta)-metric$ and the associated Riemannian metric.
- Recently some results on a class of $(\alpha,\beta)-metrics$ with constant flag curvature have been studied by Ningwei Cui, Yi-Bing Shen [5], N. Cui and Z. Lin.

II. SOME IMPORTANT DEFINITIONS

2.1 Definition: A Finsler geometry is just Riemannian geometry without the quadratic restriction.

2.2 Definition: A Finsler metric is a scalar field $L(x,y)$ which satisfies the following three conditions:

i. It is defined and differentiable at any point of $TM^n\backslash\{0\}$,
ii. It is positively homogeneous of first degree in y^i, that is, $L(x,\lambda y)=\lambda L(x,y)$, for any positive number λ,
iii. It is regular, that is, $g_{ij}(x,y)=\frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$, constitute the regular matrix g_{ij}, where $\dot{\partial}_i=\frac{\partial}{\partial y^i}$.

The manifold M^n equipped with a fundamental function $L(x,y)$ is called Finsler metric $F^n=(M^n,L)$.

2.3 Definition: Two Finsler metrics L and $\bar{L}$ are projectively related if and only if their spray coefficients have the relation

$$G^i=\bar{G}^i+P(y)y^i \quad \mathbf{(2.1)}$$

2.4 Definition : A Finsler metric is projectively related to another metric if they have the same geodesics as point sets. In Riemannian geometry, two Riemannian metrics α and $\bar{\alpha}$ are projectively related if and only if their spray coefficients have the relation

$$G^i_\alpha=G^i_{\bar{\alpha}}+\lambda_{x^k}y^k y^i \qquad \mathbf{(2.2)}$$

2.5 Definition : Let

$$\phi(s)-s\phi'(s)+(b^2-s^2)\phi''(s)>0,\quad (|s|\le b\le b_0). \qquad \mathbf{(2.3)}$$

If $\alpha=\sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta=b_iy^i$ is 1-form satisfying $||\beta_x||_\alpha<b_0\quad\forall x\in M$, then $L=\phi(s), s=\beta/\alpha$, is called an (regular) $(\alpha,\beta)-metric$. In this case, the fundamental

form of the metric tensor induced by L is positive definite.

2.6 Definition: Let $F^n=(M^n,L)$ and $\bar{F}^n=(M^n,\bar{L})$ be two Finsler spaces on a common underlying manifold M^n. If any geodesic on F^n is also a geodesic on $\bar{F}^n$ and the converse is true, then the change $L\rightarrow\bar{L}$ of the metric is called a projective change.

The relation between the geodesic coefficients G^i of L and geodesic coefficients G^i_α of α is given by

$$G^i=G^i_\alpha+\alpha Q s^i_0+\{-2Q\alpha s_0+r_{00}\}\{\Psi b^i+\Theta\alpha^{-1}y^i\}, \qquad (2.4)$$

where, $\Theta=\frac{\phi\phi'-s(\phi\phi''+\phi'\phi')}{2\phi((\phi-s\phi')+(b^2-s^2)\phi'')}$,

$$Q=\frac{\phi'}{\phi-s\phi'}, \quad \Psi=\frac{1}{2}\frac{\phi''}{(\phi-s\phi')+(b^2-s^2)\phi''}$$

2.7 Definition: Let $D^i_{jkl}=\frac{\partial^3}{\partial y^j\partial y^k\partial y^l}(G^i-\frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i)$, where G^i are the spray coefficients of L. The tensor $D=D^i_{jkl}\,\partial_i\otimes dx^j\otimes dx^k\otimes dx^l$ is called the Douglas tensor. A Finsler metric is called Douglas metric if the Douglas tensor vanishes.

Then there exists a class of scalar functions $H^i_{jk}=H^i_{jk}(x)$, such that

$$H^i_{00}=T^i-\bar{T}^i-\frac{1}{n+1}(T^m_{y^m}-\bar{T}^m_{y^m})y^i, \qquad (2.5)$$

Where $H^i_{00}=H^i_{jk}y^jy^k$, T^i and $T^m_{y^m}$ are given by the relations

$$T^i=\alpha Q s^i_0+\Psi\{-2Q\alpha s_0+r_{00}\}b^i \qquad (2.6)$$

and

$$T^m_{y^m}=Q's_0+\Psi'\alpha^{-1}(b^2-s^2)[r_{00}-2Q\alpha s_0]+2\Psi[r_0-Q'(b^2-s^2)s_0-Qss_0] \qquad (2.7)$$

III. PROJECTIVE CHANGE BETWEEN TWO FINSLER METRICS

In this section, we find the projective relation between two $(\alpha,\beta)-metrics$, that is, Special $(\alpha,\beta)-metric$ $L=\alpha+\frac{\beta^2}{\alpha}$ and Randers metric $\bar{L}=\bar{\alpha}+\bar{\beta}$ on a same underlying manifold M of dimension $n>2$.

From (2.3), $L=\alpha+\frac{\beta^2}{\alpha}$ is a regular Finsler metric if and only if 1-form β satisfies the condition $||\beta_x||_\alpha<\frac{1}{2}$ for any $x\in M$. The geodesic coefficients are given by (2.4) with ,

$$\Theta=\frac{-2s^3}{1+2b^2(1+s^2)-2s^2-3s^4}, \quad Q=\frac{2s}{1-s^2}, \quad \Psi=\frac{1}{1-3s^2+2b^2} \qquad (3.1)$$

Substituting (3.1) into (2.4), we get

$$G^i=G^i_\alpha+\frac{2\alpha^2\beta}{\alpha^2-\beta^2}s^i_0+\left\{\frac{-4\alpha^2\beta}{\alpha^2-\beta^2}s_0+r_{00}\right\}\left\{\Psi b^i+\frac{-2\alpha\beta^3}{\alpha^4+2b^2\alpha^2(\alpha^2+\beta^2)-2\alpha^2\beta^2-3\beta^4}\right\}, \qquad (3.2)$$

From (2.3), $\bar{L}=\bar{\alpha}+\bar{\beta}$ is a regular Finsler metric if and only if $||\beta_x||_\alpha<1$ for any $x\in M$.

The geodesic coefficients are given by (2.4) with

$$\bar{\Theta}=\frac{1}{2(1+s)}, \quad \bar{Q}=1, \quad \bar{\Psi}=0 \qquad (3.3)$$

First we prove the following lemma:

Lemma 3.1: Let $L=\alpha+\frac{\beta^2}{\alpha}$ and $\bar{L}=\bar{\alpha}+\bar{\beta}$ be two $(\alpha,\beta)-$metrics on a manifold M with dimension $n>2$. Then they have the same Douglas tensor if and if both the metrics L and $\bar{L}$ are Douglas metrics.

Proof: First, we prove the sufficient condition. Let L and $\bar{L}$ be Douglas metrics and corresponding Douglas tensors be D^i_{jkl} and $\bar{D}^i_{jkl}$. Then by the definition of Douglas metric, we have $D^i_{jkl}=0$ and $\bar{D}^i_{jkl}=0$, that is, both L and $\bar{L}$ have same Douglas tensor. Next, we prove the necessary condition. If L and $\bar{L}$ have the same Douglas tensor, then (2.5) holds. Substituting (3.1) and (3.3) in to (2.5), we obtain $H^i_{00}=\frac{A^i\alpha^5+B^i\alpha^4+C^i\alpha^3+D^i\alpha^2}{I\alpha^4+J\alpha^2+K}-\bar{\alpha}\bar{s}^i_0$ (3.4)

Where $A^i=(n+1)b^2(4s_0\beta-r_{00})$,

$$B^i=(n+1)\{[2(1+2b^2)s^i_0-2s_0b^i\beta+b^ir_{00}-2(1+2b^2s_0)]\},$$

$C^i = -(n+1)r_{00}(1+b^2)\beta^2$,

$D^i = -[6s_0^i\beta^3 + r_{00}b^i\beta^2 + 6s_0\beta^2]$,

$$\lambda = \frac{1}{n+1} \qquad (\mathbf{3.5})$$

and $I = (1+2b^2)(1+n)$, $J = -2\beta^2(2+b^2)(n+1)$, $K = 3\beta^4(n+1)$ $(\mathbf{3.6})$

Then (3.4) equivalent to

$$A^i\alpha^5 + B^i\alpha^4 + C^i\alpha^3 + D^i\alpha^2 = (I\alpha^4 + J\alpha^2 + K)(H_{00}^i + \bar{\alpha}\bar{s}_0^i) \quad (\mathbf{3.7})$$

Replacing y^i by $-y^i$ in (3.7) yields

$$-A^i\alpha^5 + B^i\alpha^4 - C^i\alpha^3 + D^i\alpha^2 = (I\alpha^4 - J\alpha^2 + K)(H_{00}^i - \bar{\alpha}\bar{s}_0^i) \quad (\mathbf{3.8})$$

Subtracting $(3.8)\, from$ (3.7), we obtain

$$A^i\alpha^5 + C^i\alpha^3 = H_{00}^i\alpha^2(I\alpha^2 + J) + \alpha\bar{\alpha}\bar{s}_0^i\,(K) \quad (\mathbf{3.9})$$

Now, we study two cases for Riemannian metric.

Case (i): If $\bar{\alpha} = \mu(x)\alpha$, then (3.9) reduces to

$A^i\alpha^5 + C^i\alpha^3 = H_{00}^i\alpha^2(I\alpha^2 + J) + \mu(x)\alpha^2\bar{s}_0^i(K)$ above equation can be written as

$$H^i = [H_{00}^i(I\alpha^2 + J) + \mu(x)\bar{s}_0^i(K) - A^i\alpha^5 - C^i\alpha^3]\alpha^2 \quad (\mathbf{3.11})$$

From (3.11), we can observe H^i has the factor α^2, that is, $12\lambda y^i r_{00}\beta^4$ has the factor α^2. Since β^2 is not having α^2factor, the only possibility is that βr_{00} has the factor α^2. Then for each i there exists a scalar function $\tau^i = \tau(x)$ such that $\beta r_{00} = \tau^i\alpha^2$, which is equivalent to $b_j r_{0k} + b_k r_{0j} = 2\tau^i\alpha_{jk}$.

If $n > 2$ and assuming $\tau^i \neq 0$, then $2 \geq rank\,(b_j r_{0k}) + rank(b_k r_{0j})$

$> rank\,(b_j r_{0k} + b_k r_{0j})$

$= rank(2\tau^i\alpha_{jk}) > 2$,

which is impossible unless $\tau^i = 0$. Then $\beta r_{00} = 0$. Since $\beta \neq 0$, we have $r_{00} = 0$, implies that $b_{i|j} = 0$.

Case (ii): If $\bar{\alpha} \neq \mu(x)\alpha$, then from (3.9), observe H^i has the factor α, that is, $12\lambda y^i\beta^4 r_{00}$has the factor α. Note that β^2 has no factor α. Then the only possibility is that βr_{00}has the factor α^2. As in the case(i), we have $b_{i|j} = 0$ when $n > 2$.

Special $(\alpha,\beta) - metric$ is a Douglas metric if and only if $b_{i|j} = 0$. Thus L is a Douglas metric. Since L is projectively related to $\bar{L}$, then both L and $\bar{L}$ are Douglas metrics.

Now, we prove the following theorem:

<u>Theorem 3.1</u>: The Finsler metric $L = \alpha + \beta^2/\alpha$ is projectively related to if and only if the following conditions are satisfied

$$G_\alpha^i = G_{\bar{\alpha}}^i + Py^i \ , \quad b_{i|j} = 0 \ , \quad d\bar{\beta} = 0, (\mathbf{3.12})$$

Where $b = ||\beta||_\alpha$, $b_{i|j}$ *denote the coefficients of the covariant derivatives of* β *with respect to* α, P *is a scalar function.*

Proof: Let us prove the necessary condition. Since Douglas tensor is an invariant under projective changes between two Finsler metrics, if L is projectively related to $\bar{L}$, then they have the same Douglas tensor. According to lemma 3.1 , we get both L and $\bar{L}$ are Douglas metrics.

Since Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$ is a Douglas metric if and only if $\bar{\beta}$ is closed, we have,

$$d\bar{\beta} = 0 \qquad (\mathbf{3.13})$$

and $L = \alpha + \frac{\beta^2}{\alpha}$ is a Douglas metric if and only if $b_{i|j} = 0$, $(\mathbf{3.14})$

Where $b_{i|j}$ denote the coefficients of the covariant derivatives of $\beta = b_i y^i$ with respect to α. In this case, β is closed. Hence $s_{ij} = 0$, implies that $b_{i|j} = b_{j|i}$. Thus $s_0^i = 0$, $s_0 = 0$.

By using (3.14), we have $r_{00} = r_{ij} y^i y^j = 0$. Substituting all these in (3.2) , we obtain

$$G^i = G^i_\alpha \qquad (\mathbf{3.15})$$

Since L is projective to $\bar{L}$, this is a Randers change between L and $\bar{\alpha}$. Since $\bar{\beta}$ is closed, then L is projectively related to $\bar{\alpha}$. Thus there is a scalar function $P = P(y)$ on $TM\backslash\{0\}$ such that

$$G^i = G^i_{\bar{\alpha}} + Py^i \qquad (\mathbf{3.16})$$

From (3.15) and (3.16) , we have

$$G^i_\alpha = G^i_{\bar{\alpha}} + Py^i \qquad (\mathbf{3.17})$$

Equations (3.13) and (3.14) together with (3.17) complete the proof of the necessary condition.

Since $\bar{\beta}$ is closed, it is sufficient to prove that L is projectively related to $\bar{\alpha}$. Substituting (3.14) in to (3.2) yields (3.15).

From (3.15) and (3.17) , we have

$$G^i = G^i_{\bar{\alpha}} + Py^i$$

That is, L is projectively related to $\bar{\alpha}$.
From the previous theorem, we get the following corollaries.

<u>**Corollary 3.1:**</u> The Finsler metric $L = \alpha + \frac{\beta^2}{\alpha}$ is projectively related to $\bar{L} = \bar{\alpha} + \bar{\beta}$ if and only if they are Douglas metrics and the spray coefficients of α and $\bar{\alpha}$ have the following relation

$$G^i_\alpha = G^i_{\bar{\alpha}} + Py^i$$

Where P is a scalar function.

In this, we assume that the Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$ is locally Minkowskian, where $\bar{\alpha}$ is an Euclidean metric and $\bar{\beta} = \bar{b}_i y^i$ is a one form with $\bar{b}_i = constants$. Then (3.12) can be written as

$$G^i_\alpha = Py^i \quad , \quad b_{i|j} = 0 \qquad (\mathbf{3.18})$$

Thus, we state

<u>**Corollary 3.2:**</u> The Finsler metric $L = \alpha + \frac{\beta^2}{\alpha}$ is projectivley related to $\bar{L} = \bar{\alpha} + \bar{\beta}$ if and only if L is projectively flat, in other words, L is projectively flat if and only if (3.18) holds.

CONCLUSION

1. The Finsler metric $L = \alpha + \frac{\beta^2}{\alpha}$ is projectively related to $\bar{L} = \bar{\alpha} + \bar{\beta}$ if and only if L is projectively flat, in other words, L is projectively flat if and only if $G^i_\alpha = Py^i$, $b_{i|j} = 0$ holds true.
2. The Finsler metric $L = \alpha + \frac{\beta^2}{\alpha}$ is projectively related to $\bar{L} = \bar{\alpha} + \bar{\beta}$ if and only if $G^i_\alpha = G^i_{\bar{\alpha}} + Py^i, b_{i|j} = 0$ and $d\bar{\beta} = 0$ are satisfied, where $b = ||\beta||_\alpha, b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α, P is a scalar function.
3. Let $L = \alpha + \frac{\beta^2}{\alpha}$ and $\bar{L} = \bar{\alpha} + \bar{\beta}$ be two (α, β)-metrics on a manifold M with dimension $n > 2$. Then they have the same Douglas tensor if and only if both the metrics L and $\bar{L}$ are Douglas metrics.

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