Invariant Subspaces for a Linear Transformation on a Finite Dimensional Complex Vector Space

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Abstract: In the process of advanced algebra teaching, we have learned the properties of the linear transformation on the linear space over the number field P. In this paper, we will investigated the linear transformation σ over a finite dimensional complex linear space V, we also give out the number and expression of the invariant subspaces of V respect to σ if every characteristic subspace of σ has dimension 1.

Keywords: *Complex Linear Space, Linear Transformation, Invariant Subspace, Generalized Characteristic Vector.*

Chinese Library Classification: O152.7

MSC 2010: 20G20

Let V be a linear space of dimensional n over the complex number field, σ be a linear transformation over V. If the subspace W of V satisfying that: for any $\alpha \in W$, we always have $\sigma(\alpha) \in W$, then we call W an invariant subspace of V respect to the linear transformation σ (or an σ - subspace, in short). In this paper, we will investigate the quantity of the σ - subspaces of V, we also give out the expressions of all the σ - subspaces of V if the number of them is finite.

Proposition 1 Let σ be a linear transformation of V, where V is a linear space of dimensional n over the number field F. If there exits some vector $\alpha \in V$ such that $\sigma^n(\alpha)=0$ but $\sigma^{n-1}(\alpha) \neq 0$, then V has $n+1$ σ - subspaces:

$$W_j = L(\sigma^{j-1}(\alpha), \sigma^j(\alpha), \cdots, \sigma^{n-1}(\alpha), \sigma^n(\alpha)), \quad j = 1, 2, \cdots, n+1.$$

Proof For any $1 \leq j \leq n+1$, it is easy to see that W_j is an σ - subspace. On the contrary, let W be an σ - subspace. If $W=\{0\}$, then $W = L(\sigma^n(\alpha))$. Now suppose that $W \neq \{0\}$. Since $\sigma^n(\alpha)=0$ but $\sigma^{n-1}(\alpha) \neq 0$, we know that α, $\sigma(\alpha)$, $\cdots$, $\sigma^{n-1}(\alpha)$ is a basis of V. Let $S = \{s \mid (\exists \beta \in W \setminus \{0\}) \beta = \sum_{i=s-1}^{n-1} k_i \sigma^i(\alpha),\ k_{s-1} \neq 0\}$, then $S \neq \varnothing$. Let j be the minimal number in S, then $1 \leq j \leq n$ and $W \subseteq W_j$. Next, we prove that $W_j \subseteq W$. In fact, from the minimality of j in S, we know that there exists an non-zero vector $\beta \in W$ such that

$$\beta = k_{j-1}\sigma^{j-1}(\alpha) + k_j \sigma^j(\alpha) + \cdots + k_{n-1}\sigma^{n-1}(\alpha), \quad k_{j-1} \neq 0.$$

Using σ, $\cdots$, σ^{n-j} to impact on the above equations respectively, we can get the following linear system of equations about the unknown quantities $\sigma^{j-1}(\alpha)$, $\sigma^{j}(\alpha)$, $\cdots$, $\sigma^{n-1}(\alpha)$:

$$\begin{cases} k_{j-1}\sigma^{j-1}(\alpha)+k_j\sigma^{j}(\alpha)+\cdots+k_{n-1}\sigma^{n-1}(\alpha)=\beta \\ k_{j-1}\sigma^{j}(\alpha)+\cdots+k_{n-2}\sigma^{n-1}(\alpha)=\sigma(\beta) \\ \cdots\cdots \\ k_{j-1}\sigma^{n-1}(\alpha)=\sigma^{n-j}(\beta) \end{cases}.$$

Since the determinant of the coefficient

$$D=\begin{vmatrix} k_{j-1} & k_j & \cdots & k_{n-1} \\ 0 & k_{j-1} & \cdots & k_{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & k_{j-1} \end{vmatrix}=k_{j-1}{}^{n-j+1}\neq 0,$$

by the Cramer Law, we can get that:

$$\sigma^{j-1}(\alpha)=\frac{1}{D}\begin{vmatrix} \beta & k_j & \cdots & k_{n-1} \\ \sigma(\beta) & k_{j-1} & \cdots & k_{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma^{n-j}(\beta) & 0 & \cdots & k_{j-1} \end{vmatrix}=\sum_{i=0}^{n-j} l_i\sigma^{i}(\beta)\ ,\ l_i\in F\ ,\ i=0,1,\cdots,n-j.$$

Since W is an σ - subspace, we have $\sigma(\beta),\cdots,\sigma^{n-j}(\beta)\in W$. From the above equation we can deduce that $\sigma^{j-1}(\alpha)\in W$, hence $\sigma^{j}(\alpha)$, $\cdots$, $\sigma^{n-1}(\alpha)\in W$. Therefore, $W_j=L(\sigma^{j-1}(\alpha),\sigma^{j}(\alpha),\cdots,\sigma^{n-1}(\alpha),\sigma^{n}(\alpha))\subseteq W$, as required.

Proposition 2 Let σ be a linear transformation of V, a linear space of dimensional n over the number field F. The characteristic polynomial of σ is

$$f(x)=f_1(x)f_2(x)\cdots f_r(x)\ ,$$

where $f_1(x)$, $f_2(x)$, $\cdots$, $f_r(x)$ are monic polynomials of $F[x]$ which are coprime with each other. Let $V_i=Ker(f_i(\sigma))=\{\alpha\in V\mid f_i(\sigma)\alpha=0\}$, $i=1,2,\cdots,r$. Then any σ - subspace W of V can be expressed as:

$$W=W_1\oplus W_2\oplus\cdots\oplus W_r\ ,$$

where W_i is an σ - subspace and $W_i\subseteq V_i$ fo $i=1,2,\cdots,r$.

Proof Using the same way as in proving the Theorem 12 of Chapter 7 in[1], we can prove that $V=V_1\oplus V_2\oplus\cdots\oplus V_r$, where each V_i is an σ - subspace ($i=1,2,\cdots,r$). Let W be any σ - subspace, then for any $\beta\in W$, β can be expressed uniquely as $\beta=\sum_{j=1}^{r}\beta_j$, where $\beta_j\in V_j$ for $j=1,2,\cdots,r$. For every fixed i ($1\le i\le r$), we let

$W_i=\{\beta_i \mid (\exists\beta\in W)\beta=\sum_{j=1}^{r}\beta_j,\ \beta_j\in V_j,\ \ j=1,2,\cdots,r\}\subseteq V_i$. For any $\beta_i,\ \gamma_i\in W_i$ and $k\in F$, there certainly exist $\beta,\ \gamma\in W$ such that $\beta=\sum_{j=1}^{r}\beta_j$ and $\gamma=\sum_{j=1}^{r}\gamma_j$, where $\beta_j,\ \gamma_j\in V_j$ for $j=1,2,\cdots,r$. Therefore, $k\beta+\gamma=\sum_{j=1}^{r}(k\beta_j+\gamma_j)$, $\sigma(\beta)=\sum_{j=1}^{r}\sigma(\beta_j)$. Since $\sigma(\beta)\in W$ and $k\beta_j+\gamma_j,\ \sigma(\beta_j)\in V_j$ for $j=1,2,\cdots,r$, we can deduce that $k\beta_i+\gamma_i,\ \sigma(\beta_i)\in W_i$. Therefore, W_i is an σ - subspace of V. Obviously, $W\subseteq W_1+W_2+\cdots+W_r$. On the contrary, for any $\beta_i\in W_i$ $(1\le i\le r)$, from the definition of W_i, we know that there exists $\beta\in W$ such that $\beta=\sum_{j=1}^{r}\beta_j$, where $\beta_j\in V_j$ for $j=1,2,\cdots,r$. Let $g_i(x)=\dfrac{f(x)}{f_i(x)}$, since $f_1(x)$, $f_2(x)$, $\cdots$, $f_r(x)$ are coprime with each other, we can conclude that $(f_i(x),\ g_i(x))=1$. Then there exist $u(x),\ v(x)\in F[x]$ such that $u(x)f_i(x)+v(x)g_i(x)=1$. Substitute the linear transformation σ into the expression, we can get that $u(\sigma)f_i(\sigma)+v(\sigma)g_i(\sigma)=\iota$, where ι denotes the unitary transformation over V. From $\beta_j\in V_j$ can deduce that $f_j(\sigma)\beta_j=0$ for $j=1,2,\cdots,r$. Thus $g_i(\sigma)\beta_j=0$ for $j\ne i$, $j=1,2,\cdots,r$. With the additional condition $v(\sigma)g_i(\sigma)=\iota-u(\sigma)f_i(\sigma)$, we can deduce that

$$\beta_i=\beta_i-u(\sigma)f_i(\sigma)\beta_i=(\iota-u(\sigma)f_i(\sigma))\beta_i=v(\sigma)g_i(\sigma)\beta_i$$

$$=\sum_{j=1}^{r}v(\sigma)g_i(\sigma)\beta_j=v(\sigma)g_i(\sigma)(\sum_{j=1}^{r}\beta_j)=v(\sigma)g_i(\sigma)\beta.$$

Since W is an σ - subspace and $\beta\in W$, we have $v(\sigma)g_i(\sigma)\beta\in W$, i.e., $\beta_i\in W$. Therefore, $W_i\subseteq W$ for $i=1,2,\cdots,r$. This means that $W\subseteq W_1+W_2+\cdots+W_r$, hence $W=W_1+W_2+\cdots+W_r$. Finally, from $V=V_1\oplus V_2\oplus\cdots\oplus V_r$, $W_i\subseteq V_i$, $i=1,2,\cdots,r$, we can conclude that $W_1+W_2+\cdots+W_r=W_1\oplus W_2\oplus\cdots\oplus W_r$ is a direct sum.

Conversely, for any σ - subspace W_i satisfying that $W_i\subseteq V_i$, $i=1,2,\cdots,r$, it is clear that $W_1+W_2+\cdots+W_r=W_1\oplus W_2\oplus\cdots\oplus W_r$ is an σ - subspace of V, as required.

Proposition 3 Let σ be a linear transformation of a linear space V with dimensional n over the complex field C, the characteristic polynomial of σ is

$$f(x)=(x-\lambda_1)^{n_1}(x-\lambda_2)^{n_2}\cdots(x-\lambda_r)^{n_r},$$

where $\lambda_1, \lambda_2, \cdots, \lambda_r$ are some complex numbers which are coprime with each other. $n_1, n_2, \cdots, n_r$ are positive intergers and $\sum_{i=1}^{r} n_i = n$. Denote the characteristic subspaces of σ belonging to the characteristic value $\lambda_1, \lambda_2, \cdots, \lambda_r$ are $V_{\lambda_1}, V_{\lambda_2}, \cdots, V_{\lambda_r}$ respectively. Then

(1) If there exists $1 \le i \le r$ such that $\dim V_{\lambda_i} \ge 2$, then the number of the σ-subspaces of V is infinite.

(2) If $\dim V_{\lambda_i} = 1$ for $i = 1, 2, \cdots, r$, then V has $\sum_{i=1}^{r} (n_i + 1)$ σ- subspaces:

$$W_{1,s_1} \oplus W_{2,s_2} \oplus \cdots \oplus W_{r,s_r}, \quad s_i = 1, 2, \cdots, n_i + 1, \quad i = 1, 2, \cdots, r, \qquad (*)$$

where $W_{i,s_i} = L((\sigma - \lambda_i \iota)^{s_i - 1} \alpha_i, (\sigma - \lambda_i \iota)^{s_i} \alpha_i, \cdots, (\sigma - \lambda_i \iota)^{n_i} \alpha_i)$, α_i is a generalized characteristic vector of order n_i corresponding to the characteristic value λ_i.

Proof (1) Suppose that $\dim V_{\lambda_i} \ge 2$ $(1 \le i \le r)$, then σ at least has two eigenvectors ξ_1, ξ_2 belonging to the characteristic value λ_i. For any $c \in \mathbb{C}$, let $U_c = \{\xi_1 + c\xi_2 \mid c \in \mathbb{C}\}$, then it is easy to see that U_c is an σ- subspace and satisfies that: $a \ne b$ implies that $U_a \ne U_b$. Hence the conclusion holds.

(2) Following the Theorem 12 of chapter seven in [1], $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$, where

$$V_i = Ker(\sigma - \lambda_i \iota)^{n_i} = \{\alpha \in V \mid (\sigma - \lambda_i \iota)^{n_i} \alpha = 0\}$$

is an σ-subspace, $\dim V_i = n_i$ for $i = 1, 2, \cdots, r$. For each $1 \le i \le r$, since $\dim V_{\lambda_i} = 1$, according to the Theorem 1.2.4 of [2], we know there exist $\alpha_i \in V_i$ such that $(\sigma - \lambda_i \iota)^{n_i} \alpha_i = 0$ but $(\sigma - \lambda_i \iota)^{n_i - 1} \alpha_i \ne 0$, hence α_i, $(\sigma - \lambda_i \iota)\alpha_i$, $\cdots$, $(\sigma - \lambda_i \iota)^{n_i - 1} \alpha_i$ is a set of basis of V_i. Therefore,

$$W_{i,s_i} = L((\sigma - \lambda_i \iota)^{s_i - 1} \alpha_i, (\sigma - \lambda_i \iota)^{s_i} \alpha_i, \cdots, (\sigma - \lambda_i \iota)^{n_i} \alpha_i), \quad s_i = 1, 2, \cdots, n_i + 1$$

are different subspaces of V_i. Since each W_{i,s_i} is an $(\sigma - \lambda_i \iota)$- subspace of V, $\sigma = (\sigma - \lambda_i \iota) + \lambda_i \iota$, and $\lambda_i \iota$ is a multiple transformation, we know W_{i,s_i} is also an σ- subspace of V. Therefore,

$$W_{1,s_1} \oplus W_{2,s_2} \oplus \cdots \oplus W_{r,s_r}, \quad s_i = 1, 2, \cdots, n_i + 1, \quad i = 1, 2, \cdots, r \qquad (*)$$

are $(n_1 + 1)(n_2 + 1) \cdots (n_r + 1)$ different σ- subspaces of V.

Conversely, let W be any σ- subspace of V. From Prposition 2 we know there exist σ-subspaces $U_i \subseteq V_i$, $i = 1, 2, \cdots, r$, of V such that $W = U_1 \oplus U_2 \oplus \cdots \oplus U_r$. For each $1 \le i \le r$, since V_i is an $(\sigma - \lambda_i \iota)$-subspace, we can restict $\sigma-\lambda_i\iota$ to V_i. Let $\tau_i=(\sigma-\lambda_i\iota)|_{V_i}$, then τ_i is a linear transformation of V_i and $\alpha_i\in V_i$ satisfying that $\tau_i^{n_i}\alpha_i=(\sigma-\lambda_i\iota)^{n_i}\alpha_i=0$ but $\tau_i^{n_i-1}\alpha_i=(\sigma-\lambda_i\iota)^{n_i-1}\alpha_i\neq 0$. Since U_i itself is also an $(\sigma-\lambda_i\iota)$-subspace of V and $U_i\subseteq V_i$, so U_i can be ragarded as a τ_i- subspace of V_i. From Prposition 1 , we know there exist some positive intergers $1\leq s_i\leq n_i+1$ such that $U_i=L(\tau^{s_i-1}\alpha_i,\tau^{s_i}\alpha_i,\cdots,\tau^{n_i}\alpha_i)=W_{i,s_i}$. Therefore , W is one of the σ-subspaces in (*) , as required.

Corollary Let σ be a linear transformation of V, a linear space of dimensional n over the complex field C. If σ has n different characteristic values , then V has 2^n σ- subspaces.

References

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