A Ternary Operation Related to the Complete Disjunction of Boolean Algebra

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Abstract: In this paper, we discuss about postulates, additional properties, a binary system, relation of the binary to the ternary system and realization in Boolean algebra.

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I. INTRODUCTION

In paper [3] a ternary Boolean algebra was defined as a system of elements K closed under a ternary operation $2_{(a,b,c)}$ and a unitary operation a′ satisfying universally the postulates:

(a,b, (c,d,e)) = ((a,b,c), d,(a,b,c)) --- --- --- (1.1)

(a,b,b) = (b,b,a) = b --- --- (1.2)

(a,b,b′) = (b′, b,a) = a --- --- (1.3)

For each P the elements of K form a Boolean algebra $B_{(P)}$ with the binary operations

$a\cup b=(a,p,b)$, --- --- (1.4)

$a\cap b=(a,p',b)$, --- --- --- (1.5)

under which P and P′ serve respectively as universe and null elements. In $B_{(P)}$ the ternary operation has the unique realization

$(a,b,c)=(a\cap b)\cup(b\cap c)\cup(c\cap a)$ --- --- (1.6)

Ternary systems have been discussed which are related to such binary systems as groups [2], groupoids [4] and lattices [1],[5].

1. Number is square brackets refer to the bibliography at the end of the paper.
2. In the original paper the symbol a^bc was used to denote the ternary operation here represented by (a,b,c).

   The ternary algebra posses for each pair of elements a and b an automorphism of the form:

   $f_{ab}(x)$ = [a,b,x]=((a,b,x′), (a,b′,x), (a′,b,x)), --- (1.7)

   which transforms a and b into each other. The function [a,b,x] is itself a ternary operation on the elements of K.

In the present paper we shall characterize the properties of the auto-morphism function (1.7) by suitable axioms, develop a system with two dual binary operations which have the properties of the complete disjunction operation $a+b=(a\cap b')\cup(a'\cap b)$ of Boolean rings, and show that the two systems are connected by relations analogous to (1.4) and (1.7).

II. POSTULATES

Let L be a system consisting of a set of elements closed under the ternary operation [a,b,c] and the unitary operation a′ satisfying the postulates:

[[a,b,c],d,e] = [a,b,[c,d,e]] --- --- --- (2.1)

[a,b,a] = b, --- --- --- --- (2.2)

[b,a,a′] = [a′,a,b] = b′ --- --- (2.3)

There is an element 8 such that 8′≠8 --- (2.4)

The first three postulates hold universally.

The postulate (2.1) states the basic property of the system, which we may call associativity in contrast to the distributivity of (1.1). In the sequel it will be used implicitly and we shall denote the common result of (2.1) by [a,b,c,d,e]. Postalates (2.2) and (2.3) state conditions under which the ternary operation is a function of one of the elements involved. Postulate (2.4) is included in order to exclude the trivial system in which a′=a, ∀ a. For ternary Boolean algebras having more than one element the equality of a and a′ was excluded by (1.2) and (1.3).

An immediate realization for these postulates is furnished by the automorphisms of ternary Boolean algebra. Thus the postulates are consistent.

III. ADDITIONAL PROPERTIES

We include here other simple properties of [a,b,c] which are proved from the postulates and which we shall use in this paper.

Property 3.1:(a′)′=a.

Proof: By (2.2) and (2.3), (a′)′ = [a′,a,a′]=a

Property 3.2: [a,b,b] = [b,b,a] = a

Proof:[a,b,b] = [a,b,b′,b,b′,] [by (2.2)]

= [a′,b,b′] [by (2.3)]

= a [(2.3), (3.1)]

Similarly, for the other half of the property.

Property 3.3:[a,b,a,b,c] = c

This follows from (2.2) and (3.2):

[a,b,a,b,c] = [b,b,c] = c.

Property 3.4: [a,b,c] = [a,c,b] = [b,a,c]

Proof: The proof is by (2.2) and (3.2):

[a,b,c] = [a,b,b,c,b] = [a,c,b]

= [b,a,b,b,c] = [b,a,c]

Property 3.5: For each a, $a \neq a'$

Proof: By (2.4) there is an element 8 such that $8 \neq 8'$. If a is any element of L, 8=[8,a,a] and 8′=[8,a,a′]. If a=a′, 8=8′, which is a contradiction.

Property 3.6: [a,b,c]′ = [a,b,c′] = [a′,b′,c′]

Proof:[a,b,c]′ = [a,b,c,d,d′] = [a,b,c′] by (2.3).

The rest of the theorem follows by (3.1) and (3.4)

Property 3.7: For any x, [a,b,c] = [a,x,b,x,c].

This follows by (3.2) and (3.4).

IV. A BINARY SYSTEM

Let a quasi-Boolean algebra be a set of elements Q on which are defined two binary operations (+ and ·) which satisfy the postulates:

Q is a group of involutions under the operations + and · with identities 0 and 1 respectively. --- --- (4.1)

$(a+b)\cdot c = a+(b\cdot c)$ --- --- (4.2)

there is an element a′ such that

(4.3) --- --- --- $a+a' = 1$ and $a\cdot a' = 0$, $\forall$ a.

We shall write $a+b\cdot c$ for either expression in (4.2).

A quasi-Boolean algebra has the following additional properties which are given without proof. As in Boolean algebra the duality principle holds throughout and so in general we will give only one of the two dual statements of each property. The other may be obtained by interchanging the binary operations.

Property4.4. $a+1 = a'$ (dually $a\cdot 0 = a'$)

Property4.5. The element a′ is unique $\forall$ a.

Property4.6. $a+b\cdot c = b+a\cdot c = a+c\cdot b$.

Property4.7. $a+b = a'+b'$

Property4.8. $a+b = a\cdot b'$

Property4.9. $(a+b)' = a+b'$.

As Boolean algebra, quasi-Boolean algebra has no need of either exponents or coefficients. The idempotent property is replaced by the involutionary character. Parentheses are practically superfluous by (4.2), (4.6) and (4.9). Elements involved in a expression may be interchanged almost at will (4.6). The system is not however trivial as will be seen in §6.

V. RELATION OF THE BINARY TO THE TERNARY SYSTEM

In analogy to (1.4) we may define for each P in L a binary system Q(p) with the operations

$a+b = [a,p,b]$

$a\cdot b = [a,p',b]$ --- --- (5.1)

The identities of these operations are respectively p′ and p in Q(p). We get the following theorem:

Theorem 5.1. For each p in L the system Q(p) is a quasi-Boolean algebra.

Proof: The involutionary and thus, group character of + and · follow (2.2), (2.3) and (3.4). Property (4.2) follows from (2.1) and (4.3) from (2.3) and (3.4).

In a quasi-Boolean algebra Q(p) derived from L the operation b,c has by

$[a,b,c] = [a,p,b,c] = a+b+c$

$= [a,p',b,p',c] = a\cdot b\cdot c$.

These expressions are identical by (5.5). The realization is thus unique.

The system L has automorphisms for each a and b of the from

$f_{ab}(x) = [a,b,x]$ --- --- --- --- (5.2)

From this it follows that any two binary algebra Q(p) and Q(q) derived from L are isomorphic and distinct binary algebras corresponds in a 1-1 way with abstract ternary systems. It also follows that L is homogeneous.

Theorem5.2. If p and q are any two distinct elements of L, the binary systems Q(p) and Q(q) are isomorphic.

VI. REALIZATION IN BOOLEAN ALGEBRA

Returning to the realization for [a,b,c] in terms of (a,b,c) given in (1.6) and using (1.5) we have as a realization of the ternary operation in a Boolean algebra

(6.1) --- $[a,b,c] = (a\cap b\cap c)\cup(a'\cap b'\cap c)\cup(a'\cap b\cap c')\cup(a\cap b'\cap c')$

As general realizations of the quasi-Boolean binary operations in a Boolean algebra we thus have:

$a+b = (a\cap p\cap b)\cup(a'\cap p'\cap b)\cup(a'\cap p\cap b')\cup(a\cap p'\cap b')$

$a\cdot b = (a\cap p'\cap b)\cup(a'\cap p\cap b)\cup(a\cap p\cap b')\cup(a'\cap p'\cap b')$ ------ (6.2)

A simpler realization is however obtained by assigning to p a suitable value. If we use p=0 (and thus p′=1), we obtain:

$a+b = (a'\cap b)\cup(a\cap b')$,

$a\cdot b = (a\cap b)\cup(a'\cap b')$ --- --- --- --- (6.3)

These functions are precisely the complete disjunction, the additive operation of Boolean rings and its dual function in Boolean algebra.

Theorem 6.1: The elements of a Boolean algebra form a quasi-Boolean algebra under the operation of complete disjunction and its dual. The associated ternary system is thus one that satisfies the postulates (2.1)-(2.4).

CONCLUSION

The ternary system generated by the automorphisms of a ternary Boolean algebra is thus related to the binary system induced by the complete disjunction and its dual in the same way that ternary Boolean algebra is related to Boolean algebra (cf. [3], p. 572).

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